2 Graphs of Trigonometric Functions

Objectives

Students will be able to:

- Define the nine key terms of graphs of trigonometric functions.
- Identify the period and the amplitude of trigonometric functions.
- Graph the sine and cosine function by hand.
- Identify the starting point, maximum, minimum, relative maximum, and relative minimum of a trigonometric function.
- Graph the trigonometric functions using a graphing calculator.
- Find the exact value of the inverse function of sine, cosine, and tangent.

Orienting Questions

- ✓ What are the definitions of the nine key terms in this module?
- ✓ How are the period and amplitude from the equation of a trigonometric function identified?
- ✓ What is the process of graphing the sine and cosine function by hand?
- ✓ Where is the starting point, maximum, and minimum on the graph of a trigonometric function?
- ✓ What are the steps of graphing trigonometric functions using a graphing calculator?
- ✓ How can the inverse function of sine, cosine, and tangent be found?

INTRODUCTION

The graphs of trigonometric functions allow us to model applications involving periodic behaviors. Periodic behaviors such as breathing, the pumping cycle of the human heart, tidal cycles, and alternating electrical current can be analyzed using graphs of trigonometric functions.

The goal of this module is to introduce the graphs of sine, cosine and other trigonometric functions, as well as their inverse trigonometric functions, and the applications involved with trigonometric identities.

2.1 GRAPHS OF TRIGONOMETRIC FUNCTIONS

The trigonometric functions can be graphed in a rectangular coordinate system. In this section, we will learn how to find the *x* and *y* values that will satisfy the given trigonometric function and then use the graphing calculator to verify the hand drawn graphs. Rather than using θ or t, we will use the variable x for independent variable and use y for dependent variable. In addition, the independent variable x is measured in radian for all the graphs of trigonometric function.

Before we graph, let's define some key terms. ***One Period* or *One Cycle*** of a trigonometric function is the distance that a function travels on a unit circle. The ***maximum*** is the largest y-value of the function, and the ***minimum*** is the smallest y-value of the function. A ***relative maximum*** is the largest *y*-value within a certain interval of a function, and a ***relative minimum*** is the smallest *y*-value within a certain interval of a function. An ***Amplitude***, denoted as A, of a trigonometric function is the maximum and the minimum for which the range of a trigonometric function is between $-|A|$ and $|A|$. An ***x-intercept*** is a point where the function crosses the *x*-axis.

VIDEO 2.1A

Click here to watch explanations of amplitude and period of a function by Khan Academy.

2.1.1 GRAPHS OF SINE FUNCTIONS

Let us start with the most basic sine function $y = \sin x$ (See figure 2.1.1a). Since the x-axis is measured in radian, we can pick some common radians from the unit circle for the x-values (See table 2.1.1a). Recall the trigonometric functions in terms of a unit circle from Module 1 where $\sin t = y$. Thus, the y-values in table 2.1a are the same y-values from the unit circle.

Table 2.1.1a

The *x* and the *y*-values of the graph $y = \sin x$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin x$	0	1	0	-1	0
Coordinates	$(0, 0)$	$\left(\frac{\pi}{2}, 1\right)$	$(\pi, 0)$	$\left(\frac{3\pi}{2}, -1\right)$	$(2\pi, 0)$

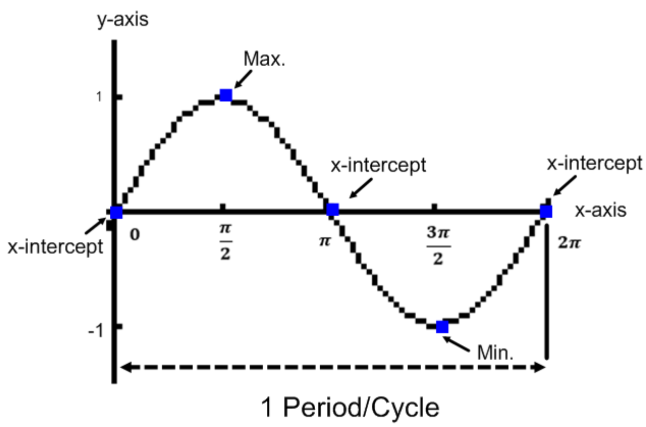


Figure 2.1.1a: The graph of sine function $y = \sin x$ in one cycle.

In figure 2.1.1a, one period of sine function is 2π. There are three x-intercepts, and the function reaches its maximum at $y = 1$ and minimum at $y = -1$. The entire cycle (period) is divided into four equal parts. The first part is where as *x* increase from 0 to $\frac{\pi}{2}$, *y* increase from 0 to 1. The second part is where as *x* increase from $\frac{\pi}{2}$ to π, *y* decrease from 1 to 0. The third part is where as *x* increase from π to $\frac{3\pi}{2}$, *y* decrease from 0 to -1. The fourth part is where as *x* increase from $\frac{3\pi}{2}$ to 2π, *y* increase from -1 to 0. We can also continuing graph figure 2.1a to obtain a more complete graph of $y = \sin x$ (See figure 2.1.1b)

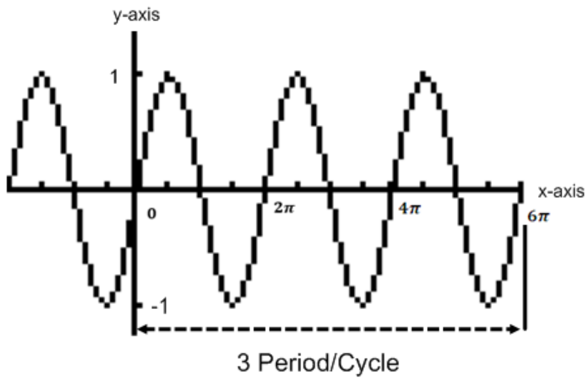


Figure 2.1.1b: A more complete graph of sine function $y = \sin x$.

To graph a various sine function, we will examine sine function in a general form. The general form of a sine function is $y = A\sin Bx$, where A is the amplitude and $Period = \frac{2\pi}{B}$. If $0 < |A| < 1$, then the sine function is shrunk vertically. If $|A| > 1$, then the sine function is stretched vertically. If $B > 1$, then the graph of sine function is shrunk horizontally. If $0 < B < 1$, then the sine function is stretched horizontally (See figure 2.1c and figure 2.1.1d).

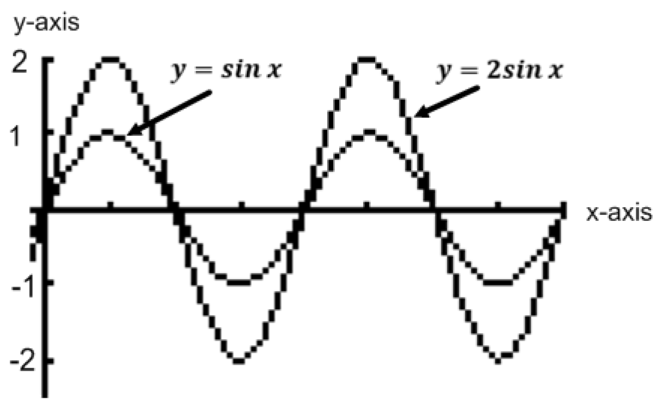


Figure 2.1.1c: The vertical stretching as compare to $y = \sin x$.

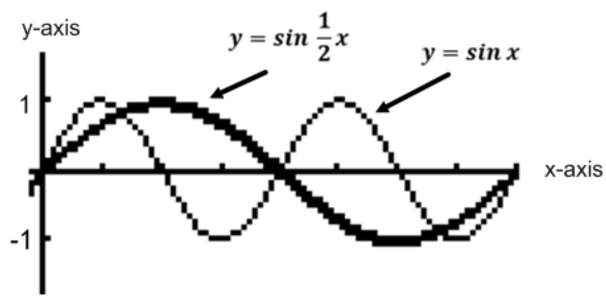


Figure 2.1.1d: The horizontal stretch as compare to $y = \sin x$.

EXAMPLES

Please work through the following examples before completing the 2.1.1 LEARNING ACTIVITY:

Example 1: Identify the amplitude and the period of $y = 2\sin x$ for $0 \le x \le 2\pi$.

The function $y = 2\sin x$ is in the form of $y = A\sin Bx$ with $A = 2$ and $B = 1$

$$Amplitude\ is |A| = |2| = 2$$

$$Period = \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi$$

Example 2: Identify the amplitude and the period of $y = \sin\frac{1}{2}x$ for $0 \le x \le 2\pi$.

The function $y = \sin\frac{1}{2}x$ is in the form of $y = A\sin Bx$ with $A = 1$ and $B = \frac{1}{2}$

$$Amplitude\ is\ |A| = |1| = 1$$

$$Period = \frac{2\pi}{\frac{1}{2}} = 2\pi \times \frac{2}{1} = 4\pi$$

Example 3: Determine whether the function $y = 2\sin 3x$ is vertical shrinking, vertical stretching, horizontal shrinking, or horizontal stretching.

The function $y = 2\sin 3x$ is in the form of $y = A\sin Bx$ with $|A| = |2| = 2 > 1$ which is vertical stretching and $B = 3 > 1$ which is horizontal shrinking.

VIDEO 2.1.1A

Click here to see examples of amplitude and period. by Khan Academy.

Since the sine function is divided into four equal parts, the *x*-values we will use to graph are the three *x*-intercepts, minimum, and the maximum. Once we find the period, start with where the function begins and add quarter periods, $\frac{Period}{4}$, to find the five key *x*-values as follows:

$x_1 = where\ the\ period\ begin,$

$$x_2 = x_1 + \frac{Period}{4},$$

$$x_3 = x_2 + \frac{Period}{4},$$

$$x_4 = x_3 + \frac{Period}{4},$$

$$x_5 = x_4 + \frac{Period}{4}$$

After determining the five *x*-values, we will substitute them into the function to find their corresponding *y*-values. Once we have found all the *x* and *y*-values, we will plot them in the rectangular coordinate system and connect them with a curve. So, the entire process to graph a sine function can be summarized as follows:

Step 1: Identify amplitude and period.

Step 2: Divide the period by 4.

Step 3: Find the five *x*-values.

Step 4: Find the corresponding y-values for the five *x*-values.

Step 5: Plot the coordinates and connect them with a curve.

EXAMPLE

Please work through the following example before completing the 2.1.1 LEARNING ACTIVITY:

Example: Graph $y = 2\sin x$ for $0 \le x \le 2\pi$.

Step 1:

The function $y = 2\sin x$ is in the form of $y = A\sin Bx$ with $A = 2$ and $B = 1$

$$\boldsymbol{Amplitude\ is |A| = |2| = 2}$$

$$\boldsymbol{Period = \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi}$$

Step 2:

$$\boldsymbol{\frac{Period}{4} = \frac{2\pi}{4} = \frac{1\pi}{2}}$$

Step 3:

$$x_1 = 0, \quad x_2 = 0 + \frac{1\pi}{2} = \frac{1\pi}{2}, \quad x_3 = \frac{1\pi}{2} + \frac{1\pi}{2} = \frac{2\pi}{2} = \pi,$$

$$x_4 = \pi + \frac{1\pi}{2} = \frac{3\pi}{2}, \quad x_5 = \frac{3\pi}{2} + \frac{1\pi}{2} = \frac{4\pi}{2} = 2\pi$$

Step 4:

$$x_1 = 0, y = 2\sin 0 = 2 \cdot 0 = 0$$

$$x_2 = \frac{1\pi}{2}, y = 2\sin\frac{1\pi}{2} = 2 \cdot 1 = 2 \quad \textit{note}: \sin\frac{1\pi}{2} = 1$$

$$x_3 = \pi, y = 2\sin\pi = 2 \cdot 0 = 0$$

$$x_4 = \frac{3\pi}{2}, y = 2\sin\frac{3\pi}{2} = 2 \cdot -1 = -2$$

$$x_5 = 2\pi, y = 2\sin 2\pi = 2 \cdot 0 = 0$$

Step 5

The coordinates are as follows:

$$(0, 0), \left(\frac{1\pi}{2}, 2\right), (\pi, 0), \left(\frac{3\pi}{2}, -2\right), (2\pi, 0)$$

Graph

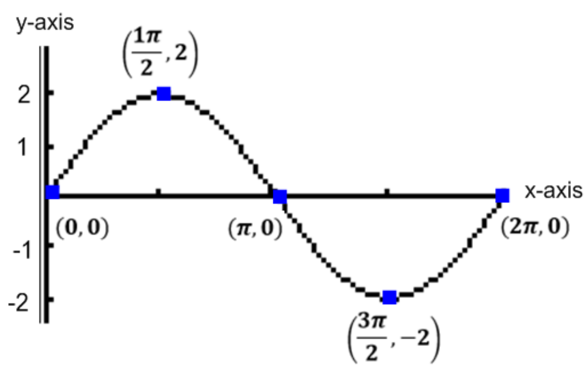


Figure 2.1.1e: The graph of $y = 2\sin x$.

VIDEO 2.1.1B

Click here to see examples of the graph of $y = 2\sin x$ for $0 \leq x \leq 2\pi$ by Ming Chan.

VIDEO2.1.1B TRANSCRIPT

Video transcript available under Module 2 in Moodle.

Other variations of the sine function involve vertical and horizontal shifts. The graph of $y = A\sin(Bx - C) + D$ determines whether the graph shift right, left, up or down. The starting point will shift $\frac{C}{B}$ units. If $\frac{C}{B}$ is postive, the function will shift right. If $\frac{C}{B}$ is negative, the function will shift left. If D is postive, the function will shift up D units. If D is negative, the function will shift down D units. Thus, the maximum will be $D + |A|$ and the minimum will be $D - |A|$ instead of $|A|$ (See figure 2.1f)

EXAMPLE

Please work through the following example before completing the 2.1.1 LEARNING ACTIVITY:

Example: Identify the starting point, maximum, and minimum of $y = 2\sin\left(2x - \frac{2\pi}{3}\right) - 3$

The function $y = 2\sin\left(2x - \frac{2\pi}{3}\right) - 3$ is in the form of $y = A\sin(Bx - C) + D$ with $A = 2$, $B = 2$, $C = \frac{2\pi}{3}$, $D = -3$

$$\boldsymbol{Amplitude\ is\ |A| = |2| = 2}$$

$$\textbf{Period} = \frac{2\pi}{2} = \pi$$

$$\textbf{Starting point is } \frac{C}{B} = \frac{\frac{2\pi}{3}}{2} = \frac{2\pi}{3} \cdot \frac{1}{2} = \frac{\pi}{3}$$

$$\textbf{Since } D = -3,$$

$$\textbf{maximum} = D + |A| = -3 + |2| = -1$$

$$\boldsymbol{minimum = D - |A| = -3 - |2| = -5}$$

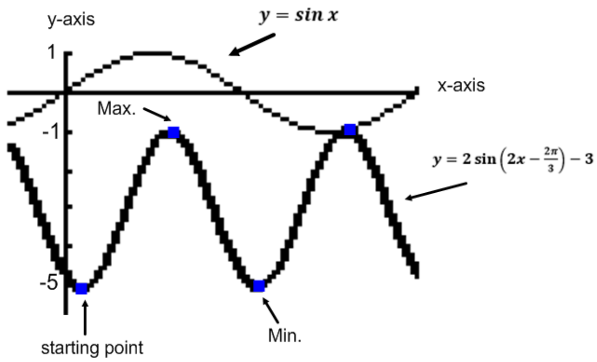


Figure 2.1.1f: vertical and horizontal shift of a sine function.

To graph sine functions using the graphing calculator, let's follow the steps below.

Step 1: Press

Step 2: Enter the sine function. Be sure to use parenthesis when entering a fraction.

Step 3: Press

Step 4: Set the appropriate viewing window.

$\text{Xmin} = \text{starting point}$

$\text{Xmax} = \text{total number of periods/cycles you wish to view.}$

$\text{Xscl} = \dfrac{Period}{4}$

$\text{Ymin} = \text{D} - |A|\ or - |A|$

$\text{Ymax} = \text{D} + |A|\ or\ |A|$

$\text{Yscl} = 1$

Step 5: Press

EXAMPLES

Please work through the following examples before completing the 2.1.1 LEARNING ACTIVITY:

Example 1: Graph $y = \sin x$ for $0 \le x \le 2\pi$ using a graphing calculator.

The function $y = \sin x$ is in the form of $y = A\sin Bx$ with $A = 1$ and $B = 1$

$$\boldsymbol{Amplitude\ is\ |A| = |1| = 1}$$

$$\boldsymbol{Period = \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi}$$

$$\boldsymbol{\frac{Period}{4} = \frac{2\pi}{4} = \frac{1\pi}{2}}$$

Step 1: **Press**

Y=

Step 2: **Enter**

SIN X,T,Θ,n)

Step 3: **Press**

WINDOW

Step 4: **Set the appropriate viewing window.**

$$\mathbf{Xmin = 0}$$

$$\mathbf{Xmax = 2\pi}$$

$$\mathbf{Xscl = \frac{\pi}{2}}$$

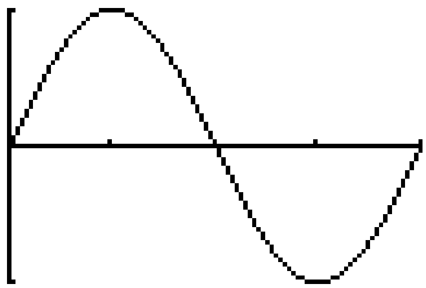
$$\mathbf{Ymin = -1}$$

$$\mathbf{Ymax = 1}$$

$$\mathbf{Yscl = 1}$$

Step 5: **Press**

GRAPH



Example 2: **Graph $y = 2\sin x$ for $0 \le x \le 2\pi$ using a graphing calculator.**

The function $y = 2\sin x$ is in the form of $y = A\sin Bx$ with $A = 2$ and $B = 1$

$$\boldsymbol{Amplitude\ is\ |A| = |2| = 2}$$

$$\boldsymbol{Period = \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi}$$

$$\boldsymbol{\frac{Period}{4} = \frac{2\pi}{4} = \frac{\pi}{2}}$$

Step 1: **Press**

Y=

Step 2: **Enter**

2 SIN X,T,θ,n)

Step 3: **Press**

WINDOW

Step 4: **Set the appropriate viewing window.**

$$\mathbf{Xmin} = \mathbf{0}$$

$$\mathbf{Xmax} = \mathbf{2\pi}$$

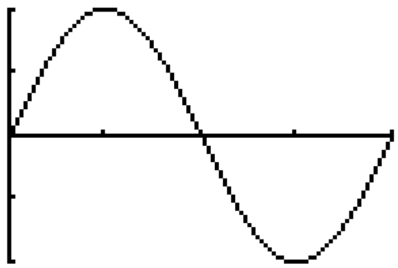
$$\mathbf{Xscl} = \boldsymbol{\frac{\pi}{2}}$$

$$\mathbf{Ymin} = \mathbf{-2}$$

$\mathbf{Ymax = 2}$

$\mathbf{Yscl = 1}$

Step 5: **Press**



Example 3: Graph one period of $y = 2\sin\left(2x - \frac{2\pi}{3}\right) - 3$

The function $y = 2\sin\left(2x - \frac{2\pi}{3}\right) - 3$ **is in the form of** $y = A\sin(Bx - C) + D$ **with** $A = 2$, $B = 2$, $C = \frac{2\pi}{3}$, $D = -3$

$$Amplitude\ is\ |A| = |2| = 2$$

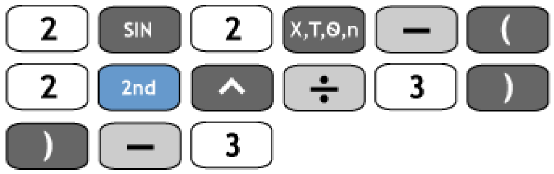
$$\text{Period} = \frac{2\pi}{2} = \pi$$

$$\text{Starting point is } \frac{C}{B} = \frac{\frac{2\pi}{3}}{2} = \frac{2\pi}{3} \cdot \frac{1}{2} = \frac{\pi}{3}$$

$$\frac{Period}{4} = \frac{\pi}{4} = \frac{\pi}{4}$$

Step 1: **Press**

Step 2: **Enter**



Step 3: **Press**

Step 4: **Set the appropriate viewing window.**

$$\mathbf{Xmin} = \frac{\pi}{3}$$

$$\mathbf{Xmax} = \frac{4\pi}{3} \;\; \textbf{since the starting point is shifted to } \frac{\pi}{3}, \textbf{add } \frac{\pi}{3} \textbf{ to } \pi$$

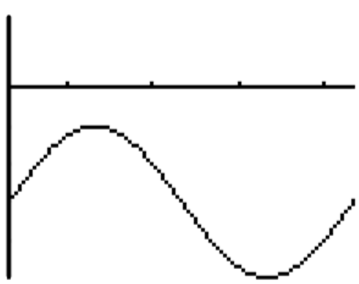
$$\mathbf{Xscl} = \frac{\pi}{4}$$

$$\mathbf{Ymin} = -3 - 2 = -5$$

$$\mathbf{Ymax} = -3 + 2 = -1 \;\; \textbf{we are going to set it was 1, so we can see the x} - \textbf{axis}$$

$$\mathbf{Yscl} = 1$$

Step 5: **Press**



2.1.1 LEARNING ACTIVITY

a. Identify the amplitude and the period of $y = 3\sin x$ for $0 \leq x \leq 2\pi$.

b. Identify the amplitude and the period of $y = 2\sin\frac{1}{3}x$ for $0 \leq x \leq 6\pi$.

c. Determine whether the function is vertical shrinking, vertical stretching, horizontal shrinking, or horizontal stretching for $y = 3\sin\frac{1}{2}x$.

d. Graph $y = 3\sin 2x$ for $0 \leq x \leq 2\pi$.

e. Identify the starting point, maximum, and minimum of $y = 2\sin\left(2x - \frac{\pi}{2}\right) + 1$.

f. Graph $y = 2\sin\left(2x - \frac{\pi}{2}\right) + 1$using a graphing calculator.

2.1.2 GRAPHS OF COSINE FUNCTIONS

The basic cosine function is $y = \cos x$ (See figure 2.2a). We can use the same x-values as in Table 2.1.2a to find the corresponding y-values (See table 2.1.2a). Recall the trigonometric functions in terms of a unit circle from module one where $\cos t = x$. Thus, the y-values in table 2.2a are the same y-values from unit circle.

Table 2.1.2a

The *x* and the *y*-values of the graph $y = \cos x$

x	**0**	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos x$	**1**	**0**	**-1**	**0**	**1**
Coordinates	$(0, 1)$	$\left(\frac{\pi}{2}, 0\right)$	$(\pi, -1)$	$\left(\frac{3\pi}{2}, 0\right)$	$(2\pi, 1)$

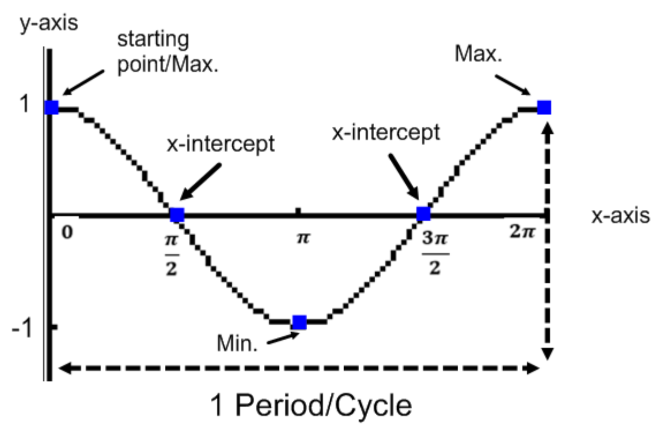


Figure 2.1.2a: The graph of cosine function $y = \cos x$ in one cycle.

In figure 2.1.2a, one period of cosine function is 2π. The graph has two x-intercepts at $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$, maximum at $(0,1)$ and $(2\pi, 1)$, minimum at $(0, -1)$. The entire cycle (period) is divided into four equal parts. The first part is where as *x* increase from 0 to $\frac{\pi}{2}$, *y* decreases from 1 to 0. The second part is where as *x* increase from $\frac{\pi}{2}$ to π, *y* decrease from 0 to -1. The third part is where as *x* increase from π to $\frac{3\pi}{2}$, *y* increase from -1 to 0. The fourth part is where as *x* increase from $\frac{3\pi}{2}$ to 2π, *y* increase from 0 to 1.

The general form of a cosine function is $y = A\cos(Bx - C) + D$, where A is the amplitude and $\text{Period} = \frac{2\pi}{B}$. If $0 < |A| < 1$, then the function is shrunk vertically. If $|A| > 1$, then the function is stretched vertically. If $B > 1$, then the function is shrunk horizontally. If $0 < B < 1$, then the function is stretched horizontally. The starting point will shift $\frac{C}{B}$ units. If $\frac{C}{B}$ is postive, the function will shift right. If $\frac{C}{B}$ is negative, the function will shift left. D determines the vertical shift. If D is postive, the function will shift up D units. If D is negative, the function will shift down D units. Thus, the maximum will be $D + |A|$ and the minimum will be $D - |A|$ instead of $|A|$.

EXAMPLES

Please work through the following examples before completing the 2.1.2 LEARNING ACTIVITY:

Example 1: Identify the amplitude and the period of $y = 2\cos 3x$ for $0 \leq x \leq 2\pi$.

The function $y = 2\cos 3x$ is in the form of $y = A\cos(Bx - C) + D$ with $A = 2$, $B = 3$, $C = 0$, and $D = 0$.

$$\boldsymbol{Amplitude\ is |A| = |2| = 2}$$

$$\boldsymbol{Period = \frac{2\pi}{B} = \frac{2\pi}{3}}$$

Example 2: Determine whether the function $y = 2\cos 3x$ is vertical shrinking, vertical stretching, horizontal shrinking, or horizontal stretching.

The function $y = 2\cos 3x$ is in the form of $y = A\cos Bx$ with $|A| = |2| = 2 > 1$ which is vertical stretching and $B = 3 > 1$ which is horizontal shrinking.

Since the cosine function is divided into four equal parts, the *x*-values we will use to graph are the two *x*-intercepts, minimum, and the maximum. Once we find the period, start with where the function begins and add quarter periods, $\frac{Period}{4}$, to find the five *x*-values as follows:

$$x_1 = where\ the\ period\ begin,$$

$$x_2 = x_1 + \frac{Period}{4},$$

$$x_3 = x_2 + \frac{Period}{4},$$

$$x_4 = x_3 + \frac{Period}{4},$$

$$x_5 = x_4 + \frac{Period}{4}$$

After determining the five *x*-values, we will substitute them into the function to find their corresponding *y*-values. Once we have found all the *x* and *y*-values, we will plot them in the rectangular coordinate system and connect them with a curve. So, the entire process to graph a cosine function can be summarized as follows:

Step 1: Identify amplitude and period.

Step 2: Divide the period by 4.

Step 3: Find the five *x*-values.

Step 4: Find the corresponding y-values for the five *x*-values.

Step 5: Plot the coordinates and connect them with a curve.

EXAMPLES

Please work through the following examples before completing the 2.1.2 LEARNING ACTIVITY:

Example 1: Graph one period of $y = 2\cos 3x$.

Step 1:

The function $y = 2\cos 3x$ is in the form of $y = A\cos(Bx - C) + D$ with $A = 2$, $B = 3$, $C = 0$, and $D = 0$.

$$Amplitude\ is|A| = |2| = 2$$

$$Period = \frac{2\pi}{B} = \frac{2\pi}{3}$$

Since $D = 0$,

$$\text{maximum} = D + |A| = 0 + |2| = 2$$

$$minimum = D - |A| = 0 - |2| = -2$$

Step 2:

$$\frac{Period}{4} = \frac{\frac{2\pi}{3}}{4} = \frac{\pi}{6}$$

Step 3:

$$\text{Starting point is } \frac{C}{B} = \frac{0}{3} = 0$$

$$x_1 = 0, \qquad x_2 = 0 + \frac{\pi}{6} = \frac{\pi}{6}, \qquad x_3 = \frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3},$$

$$x_4 = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}, \qquad x_5 = \frac{\pi}{2} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

Step 4:

$$x_1 = 0, y = 2\cos 3 \cdot 0 = 2\cos 0 = 2 \cdot 1 = 2 \quad \textit{note}\text{: } \cos 0 \text{ or } \cos 0° = 1$$

$$x_2 = \frac{\pi}{6}, y = 2\cos 3 \cdot \frac{\pi}{6} = 2\cos\frac{\pi}{2} = 2 \cdot 0 = 0 \quad \textit{note}\text{: } \cos\frac{\pi}{2} \text{ or } \cos 90° = 0$$

$$x_3 = \frac{\pi}{3}, y = 2\cos 3 \cdot \frac{\pi}{3} = 2\cos\pi = 2 \cdot -1 = -2 \quad \textit{note}\text{: } \cos\pi \text{ or } \cos 180° = -1$$

$$x_4 = \frac{\pi}{2}, y = 2\cos 3 \cdot \frac{\pi}{2} = 2\cos\frac{3\pi}{2} = 2 \cdot 0 = 0 \quad \textit{note}\text{: } \cos\frac{3\pi}{2} \text{ or } \cos 270° = 0$$

$$x_5 = \frac{2\pi}{3}, y = 2\cos 3 \cdot \frac{2\pi}{3} = 2\cos 2\pi = 2 \cdot 1 = 2$$

Step 5:

The coordinates are as follows:

$$(0, 2), \left(\frac{\pi}{6}, 0\right), \left(\frac{\pi}{3}, -2\right), \left(\frac{\pi}{2}, 0\right), \left(\frac{2\pi}{3}, 2\right)$$

Graph

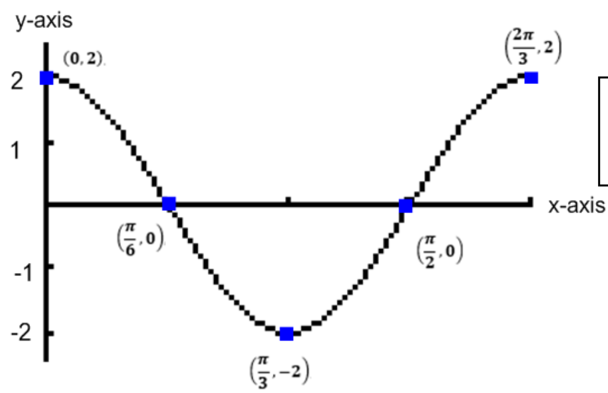


Figure 2.1.2b: The graph of $y = 2\cos 3x$ in one cycle.

Example 2: Graph one period of $y = -2\cos 3x$.

Step 1:

The function $y = 2\cos 3x$ is in the form of $y = A\cos(Bx - C) + D$ with $A = -2$, $B = 3$, $C = 0$, and $D = 0$.

$$Amplitude\ is |A| = |-2| = 2$$

$$Period = \frac{2\pi}{B} = \frac{2\pi}{3}$$

Since $D = 0$,

$$\text{maximum} = D + |A| = 0 + |2| = 2$$

$$minimum = D - |A| = 0 - |2| = -2$$

Step 2:

$$\frac{Period}{4} = \frac{\frac{2\pi}{3}}{4} = \frac{\pi}{6}$$

Step 3:

$$\text{Starting point is } \frac{C}{B} = \frac{0}{3} = 0$$

$$x_1 = 0, \quad x_2 = 0 + \frac{\pi}{6} = \frac{\pi}{6}, \quad x_3 = \frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6} = \frac{\pi}{3},$$

$$x_4 = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}, \quad x_5 = \frac{\pi}{2} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

Step 4:

$$x_1 = 0, y = -2\cos 3 \cdot 0 = -2\cos 0 = -2 \cdot 1 = -2 \quad note\text{: } \cos 0 \text{ or } \cos 0° = 1$$

$$x_2 = \frac{\pi}{6}, y = -2\cos 3 \cdot \frac{\pi}{6} = -2\cos\frac{\pi}{2} = -2 \cdot 0 = 0 \quad note\text{: } \cos\frac{\pi}{2} \text{ or } \cos 90° = 0$$

$$x_3 = \frac{\pi}{3}, y = -2\cos 3 \cdot \frac{\pi}{3} = -2\cos\pi = -2 \cdot -1 = 2 \quad note\text{: } \cos\pi \text{ or } \cos 180° = -1$$

$$x_4 = \frac{\pi}{2}, y = -2\cos 3 \cdot \frac{\pi}{2} = -2\cos\frac{3\pi}{2} = -2 \cdot 0 = 0 \quad note\text{: } \cos\frac{3\pi}{2} \text{ or } \cos 270° = 0$$

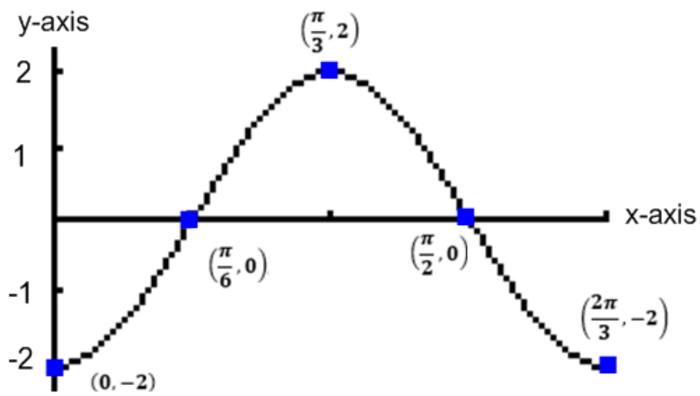
$$x_5 = \frac{2\pi}{3}, y = -2\cos 3 \cdot \frac{2\pi}{3} = -2\cos 2\pi = -2 \cdot 1 = -2$$

Step 5:

The coordinates are as follows:

$$(\mathbf{0},-\mathbf{2}),\left(\frac{\pi}{6},\mathbf{0}\right),\left(\frac{\pi}{3},\mathbf{2}\right),\left(\frac{\pi}{2},\mathbf{0}\right),\left(\frac{2\pi}{3},-\mathbf{2}\right)$$

Graph



To graph cosine functions using the graphing calculator, let's follow the steps below.

Step 1: Press

Step 2: Enter the cosine function. Be sure to use parenthesis when entering a fraction.

Step 3: Press

Step 4: Set the appropriate viewing window.

$\text{Xmin} = \text{starting point}$

$\text{Xmax} = \text{total number of periods/cycles you wish to view.}$

$$\text{Xscl} = \frac{Period}{4}$$

$\text{Ymin} = \text{D} - |A| \ or - |A|$

$\text{Ymax} = \text{D} + |A| \ or \ |A|$

$\text{Yscl} = 1$

Step 5: Press

GRAPH

EXAMPLES

<u>*Please work through the following examples before completing the 2.1.2 LEARNING ACTIVITY:*</u>

Example 1: Use the graphing calculator to graph one period of $y = 2\cos 3x$.

The function $y = 2\cos 3x$ is in the form of $y = A\cos(Bx - C) + D$ with $A = 2$, $B = 3$, $C = 0$, and $D = 0$.

$$\boldsymbol{Amplitude\ is |A| = |2| = 2}$$

$$\boldsymbol{Period = \frac{2\pi}{B} = \frac{2\pi}{3}}$$

$$\textbf{maximum} = \boldsymbol{D + |A| = 0 + |2| = 2}$$

$$\boldsymbol{minimum = D - |A| = 0 - |2| = -2}$$

$$\boldsymbol{\frac{Period}{4} = \frac{\frac{2\pi}{3}}{4} = \frac{\pi}{6}}$$

$$\textbf{Starting point is } \boldsymbol{\frac{C}{B} = \frac{0}{3} = 0}$$

Step 1: **Press**

Y=

Step 2: **Enter**

2 COS (3 X,T,Θ,n)

Step 3: **Press**

WINDOW

Step 4: **Set the appropriate viewing window.**

$$\mathbf{Xmin} = \mathbf{0}$$

$$\mathbf{Xmax} = \frac{\mathbf{2\pi}}{\mathbf{3}}$$

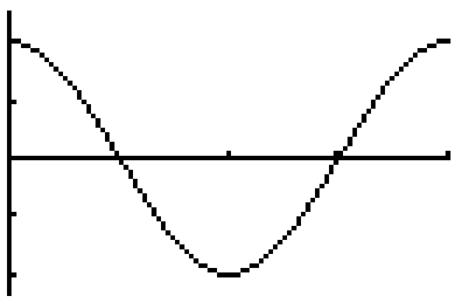
$$\mathbf{Xscl} = \frac{\boldsymbol{\pi}}{\mathbf{6}}$$

$$\mathbf{Ymin} = -\mathbf{2}$$

$$\mathbf{Ymax} = \mathbf{2}$$

$$\mathbf{Yscl} = \mathbf{1}$$

Step 5: **Press**



2.1.2 LEARNING ACTIVITY

a. Identify the amplitude and the period of $y = -2\cos x$ for $0 \le x \le 2\pi$.

b. Determine whether the function $y = 3\cos\frac{1}{3}x$ is vertical shrinking, vertical stretching, horizontal shrinking, or horizontal stretching.

c. Graph one period of $y = 3\cos\frac{1}{3}x$.

d. Use the graphing calculator to graph one period of $y = 3\cos\frac{1}{3}x$.

2.1.3 Graphs Of Tangent And Cotangent Functions

The graphs of $y = \tan x$ is significantly different from sine and cosine because of its properties. The properties of tangent function are $\text{Period} = \pi$, vertical asymptotes at every odd multiples of $\frac{\pi}{2}$, and the *x*-intercepts are midway between a pair of vertical asymptotes.

The various graph of tangent function has the form $y = A\tan(Bx - C)$. To graph a tangent function, we will find a pair of consecutive vertical asymptote which is $-\frac{\pi}{2} < Bx - c < \frac{\pi}{2}$. The *x*-intercepts are midway between a pair of vertical asymptotes (See figure 2.1.3).

To graph tangent functions using the graphing calculator, let's follow the steps below.

Step 1: Press

Y=

Step 2: Enter the tangent function. Be sure to use parenthesis when entering a fraction.

Step 3: Press

WINDOW

Step 4: Set the appropriate viewing window.

$\text{Xmin} = \text{first consecutive vertical asymptote for the number of periods you wish to view}$

$\text{Xmax} = \text{last consecutive vertical asymptote for the number of periods you wish to view.}$

$\text{Xscl} = \text{distance between the first pair consecutive asymptotes divided by } 4$

$\text{Ymin} = \text{a little smaller} - |A| \quad \textit{since the range is } (-\infty, \infty)$

$\text{Ymax} = \text{a little bigger than } |A| \quad \textit{since the range is } (-\infty, \infty)$

$\text{Yscl} = 1$

Step 5: Press

GRAPH

EXAMPLES

Please work through the following examples before completing the 2.1.3 LEARNING ACTIVITY:

Example 1: Use the graphing calculator to graph two period of $y = \tan\frac{1}{2}x$.

The function $y = \tan\frac{1}{2}x$ is in the form of $y = A\tan(Bx - C)$ with $A = 1$, $B = \frac{1}{2}$, $C = 0$. The first pair consecutive vertical asymptotes are

$$-\frac{\pi}{2} < \frac{1}{2}x - 0 < \frac{\pi}{2} \quad \textit{Multiply all parts by 2}$$

$$-\pi < x < \pi$$

The second pair consecutive vertical asymptotes are at every odd multiples of $\frac{\pi}{2}$,

$$\frac{\pi}{2} < \frac{1}{2}x - 0 < \frac{3\pi}{2} \quad \textit{Multiply all parts by 2}$$

$$\pi < x < 3\pi$$

The *x*-intercepts are midway between two consective asymptote,

$$first\ x - int. = \frac{-\pi + \pi}{2} = 0$$

$$second\ x - int. = \frac{\pi + 3\pi}{2} = 2\pi$$

***Step 1:* Press**

Y=

***Step 2:* Enter**

TAN (1 ÷ 2)
X,T,Θ,n)

***Step 3:* Press**

WINDOW

Step 4: **Set the appropriate viewing window.**

$\mathbf{Xmin = -\pi}$

$\mathbf{Xmax = 3\pi}$

$\mathbf{Xscl = \pi}$

$\mathbf{Ymin = -5}$

$\mathbf{Ymax = 5}$

$\mathbf{Yscl = 1}$

Step 5: **Press**

GRAPH

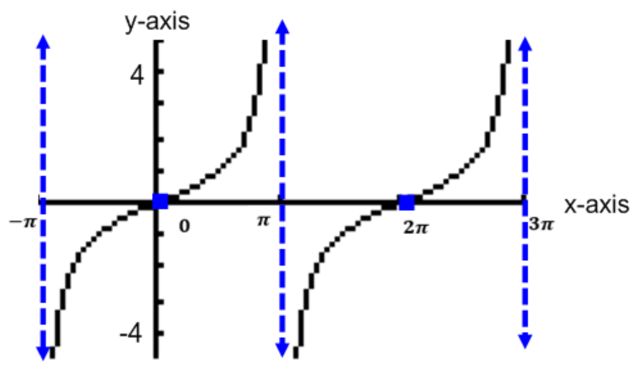


Figure 2.1.3a: The graph of $y = \tan\frac{1}{2}x$ in two periods.

Example 2: Use the graphing calculator to graph two period of $y = \tan\left(x + \frac{\pi}{4}\right)$.

The function $y = \tan\left(x + \frac{\pi}{4}\right)$ is in the form of $y = A\tan\left(Bx - C\right)$ with $A = 1$, $B = 1$, $C = -\frac{\pi}{4}$. The two vertical asymptotes are

$\boldsymbol{first\ pair\ of\ vertical\ asymptote}$:

$$-\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2} \quad \boldsymbol{subtract\ all\ parts\ by\ \frac{\pi}{4}}$$

$$-\frac{3\pi}{4} < x < \frac{\pi}{4}$$

$\boldsymbol{second\ pair\ of\ vertical\ asymptote}$:

$$\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{3\pi}{2} \quad \textit{subtract all parts by } \frac{\pi}{4}$$

$$\frac{\pi}{4} < x < \frac{5\pi}{4}$$

The *x*-intercepts are midway between two consective asymptote,

$$first\ x - int. = \frac{\frac{-3\pi}{4} + \frac{\pi}{4}}{2} = \frac{-\frac{2\pi}{4}}{2} = -\frac{\pi}{4}$$

$$second\ x - int. = \frac{\frac{\pi}{4} + \frac{5\pi}{4}}{2} = \frac{\frac{6\pi}{4}}{2} = \frac{5\pi}{4}$$

Step 1: **Press**

Y=

Step 2: **Enter**

TAN X,T,Θ,n + (2nd ^
÷ 4))

Step 3: **Press**

WINDOW

Step 4: **Set the appropriate viewing window.**

$$\text{Xmin} = -\frac{3\pi}{4}$$

$$\text{Xmax} = \frac{5\pi}{4}$$

$$\text{Xscl} = \frac{\pi}{4}$$

$$\text{Ymin} = -5$$

$$\text{Ymax} = 5$$

$$\text{Yscl} = 1$$

***Step 5:* Press**

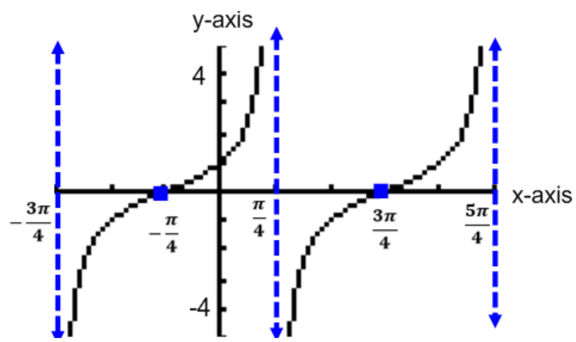


Figure 2.1.3b: The graph of $y = \tan\left(x + \frac{\pi}{4}\right)$ in two periods.

Similar to tangent function, the graph of $y = \cot x$ also has period of π. The function has vertical asymptote at every interval of π. The *x*-intercepts occur at half way between two consecutive vertical asymptotes. The general form of cotangent function is $y = A\cot\,(Bx - C)$. To find two consecutive vertical asymptote in one period, we will set it up as $0 < Bx - C < \pi$.

Since the graphing calculatgor does not contains a cotangent function key, we will use the reciprocal identity of tangent, which is $\cot x = \frac{1}{\tan x}$, to graph cotangent in the calculator. To graph cotangent functions using the graphing calculator, let's follow the steps below.

Step 1: Press

Step 2: Enter the cotangent function. Be sure to use parenthesis when entering a fraction.

Step 3: Press

Step 4: Set the appropriate viewing window.

$\text{Xmin} = \text{first consecutive vertical asymptote for the number of periods you wish to view}$

$\text{Xmax} = \text{last consecutive vertical asymptote for the number of periods you wish to view.}$

$\text{Xscl} = \text{distance between the first consecutive asymptotes divided by } 4$

$\text{Ymin} = \text{a little smaller than } -|A| \quad \textit{since the range is } (-\infty, \infty)$

$\text{Ymax} = \text{a greater than } |A| \quad \textit{since the range is } (-\infty, \infty)$

$\text{Yscl} = 1$

Step 5: Press

EXAMPLES

Please work through the following examples before completing the 2.1.3 LEARNING ACTIVITY:

Example 1: Use the graphing calculator to graph two period of $y = \cot 2x$.

The function $y = \cot 2x$ is in the form of $y = A\cot\,(Bx - C)$ with $A = 1$, $B = 2$, $C = 0$. The two pair consecutive vertical asymptotes are

$\boldsymbol{first\ pair\ of\ vertical\ asymptote}$:

$\mathbf{0} < 2x - \mathbf{0} < \pi \quad \textbf{\textit{divide all parts by 2}}$

$$\mathbf{0} < x < \frac{\pi}{2}$$

$\boldsymbol{Second\ pair\ of\ vertical\ asymptote}$:

$\pi < 2x - \mathbf{0} < 2\pi \quad \textbf{\textit{divide all parts by 2}}$

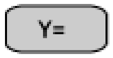
$$\frac{\pi}{2} < x < \pi$$

The *x*-intercepts are midway between two consective asymptote,

$$\boldsymbol{First\ x - int.} = \frac{0 + \frac{\pi}{2}}{2} = \frac{\pi}{2} \times \frac{1}{2} = \frac{\pi}{4}$$

$$\boldsymbol{Second\ x - int.} = \frac{\frac{\pi}{2} + \pi}{2} = \frac{3\pi}{2} \times \frac{1}{2} = \frac{3\pi}{4}$$

***Step 1:* Press**



Step 2: **Enter**

Step 3: **Press**

Step 4: **Set the appropriate viewing window.**

$\mathbf{Xmin = 0}$

$\mathbf{Xmax = \pi}$

$\mathbf{Xscl = \frac{\pi}{8}}$

$\mathbf{Ymin = -5}$

$\mathbf{Ymax = 5}$

$\mathbf{Yscl = 1}$

Step 5: **Press**

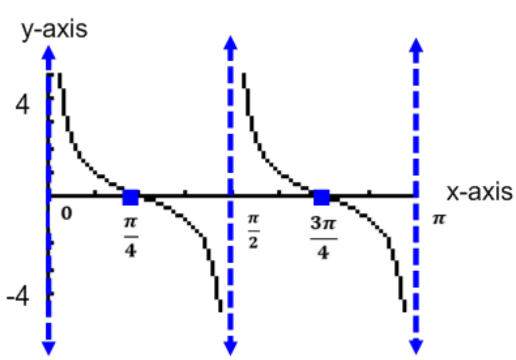


Figure 2.1.3c: The graph of $y = \cot 2x$ in two periods.

2.1.3 LEARNING ACTIVITY

a. Use the graphing calculator to graph two period of $y = \tan\frac{1}{3}x$.

b. Use the graphing calculator to graph two period of $y = \tan\left(x - \frac{\pi}{4}\right)$.

c. Use the graphing calculator to graph two period of $y = \cot 3x$.

2.1.4 Graphs Of Cosecant And Secant Functions

To graph cosecant and secant functions, we will use the reciprocal identity. Since

$\csc x = \frac{1}{\sin x}$ and $\sec x = \frac{1}{\cos x}$, we will take the reciprocal of the *y*-values from the table and use these as the *y*-values for $y = \csc x$ and $y = \sec x$ (See table 2.1.3a and 2.1.3b). In table 2.1.3a and 2.1.3b, if the *y*-values for $y = \sin x$ and $y = \cos x$ are zero, then cosecant and secant are undefined at the corresponding *x*-values. Thus, the vertical asymptotes will occur at these *x*-values.

Table 2.1.4a

The *x* and the *y*-values of the graph $y = \csc x$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \sin x$	0	1	0	-1	0
$y = \csc x$	$\frac{1}{0}$	$\frac{1}{1}$	$\frac{1}{0}$	$\frac{1}{-1}$	$\frac{1}{0}$
Coordinates	**undefined**	$\left(\frac{\pi}{2}, 1\right)$	**undefined**	$\left(\frac{3\pi}{2}, -1\right)$	**undefined**

Table 2.1.4b

The *x* and the *y*-values of the graph $y = \sec x$

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$y = \cos x$	1	0	-1	0	1
$y = \sec x$	1	$\frac{1}{0}$	$\frac{1}{-1}$	$\frac{1}{0}$	$\frac{1}{1}$
Coordinates	$(0, 1)$	**undefined**	$(\pi, -1)$	**undefined**	$(2\pi, 1)$

Both cosecant and secant functions have general form $y = A\csc(Bx - C) + D$ and $y = A\sec(Bx - C) + D$. The vertical asymptotes for cosecant function occurs at the multiples of π, and the vertical asymptotes for secant function occur at the odd multiples of $\frac{\pi}{2}$. The length of the period is $\frac{2\pi}{B}$. The starting point will shift $\frac{C}{B}$ units from the origin. The relative maximum is $D + |A|$ and the minimum is $D - |A|$. If $0 < |A| < 1$, then the function is shrunk vertically. If $|A| > 1$, then the function is stretched vertically. If $B > 1$, then the function is shrunk horizontally. If $0 < B < 1$, then the function is stretched horizontally.

To graph cosecant and secant functions using the graphing calculator, let's follow the steps below.

Step 1: Press

Y=

Step 2: Enter the cosecant or secant function. Be sure to use parenthesis when entering a fraction.

Step 3: Press

WINDOW

Step 4: Set the appropriate viewing window.

$\text{Xmin} = \text{starting point}$

$\text{Xmax} = \text{total number of periods/cycles you wish to view.}$

$$\text{Xscl} = \frac{Period}{4}$$

$\text{Ymin} = \text{relative minimum, a little less than D} - |A|\ or - |A| \quad since\ the\ range\ is\ (-\infty, \infty)$

$\text{Ymax} = \text{relative maximum, a little greater than}\ \text{D} + |A|\ or\ |A| \quad since\ the\ range\ is\ (-\infty, \infty)$

$\text{Yscl} = 1$

Step 5: Press

GRAPH

EXAMPLES

Please work through the following examples before completing the 2.1.4 LEARNING ACTIVITY:

Example 1: Use the graphing calculator to graph one period of $y = 2\csc 2x$.

The function $y = 2\csc 2x$ is in the form of $y = A\csc(Bx - C) + D$ with $A = 2$, $B = 2$, $C = 0$, $D = 0$.

$$Amplitude\ is|A| = |2| = 2$$

$$Period = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

$$Starting\ point = \frac{C}{B} = \frac{0}{2} = 0$$

$$\frac{Period}{4} = \frac{\pi}{4}$$

The five x-values are $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and π. Since $\csc x = \frac{1}{\sin x}$, we will view $y = 2\csc 2x$ as $y = \frac{2}{\sin 2x}$.

$$x = 0, y = \frac{2}{\sin 2 \cdot 0} = \frac{2}{\sin 0} = \frac{2}{0} = undefined$$

$$x = \frac{\pi}{4}, y = \frac{2}{\sin 2 \cdot \frac{\pi}{4}} = \frac{2}{\sin \frac{\pi}{2}} = \frac{2}{1} = 2$$

$$x = \frac{\pi}{2}, y = \frac{2}{\sin 2 \cdot \frac{\pi}{2}} = \frac{2}{\sin \pi} = \frac{2}{0} = undefined$$

$$x = \frac{3\pi}{4}, y = \frac{2}{\sin 2 \cdot \frac{3\pi}{4}} = \frac{2}{\sin \frac{3\pi}{2}} = \frac{2}{-1} = -2$$

$$x = \pi, y = \frac{2}{\sin 2 \cdot \pi} = \frac{2}{\sin 2\pi} = \frac{2}{0} = undefined$$

The x and the y-values of the graph $y = 2\csc 2x$					
x	**0**	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$y = \frac{2}{\sin 2x}$	**undefined**	**2**	**undefined**	**−2**	**undefined**
Coordinates	**undefined**	$\left(\frac{\pi}{4}, 2\right)$	**undefined**	$\left(\frac{3\pi}{4}, -2\right)$	**undefined**

Step 1: **Press**

Y=

Step 2: **Enter**

2 ÷ (SIN 2 X,T,Θ,n
)

Step 3: **Press**

WINDOW

Step 4: **Set the appropriate viewing window.**

$\mathbf{Xmin = 0}$

$\mathbf{Xmax = \pi}$

$\mathbf{Xscl = \frac{\pi}{4}}$

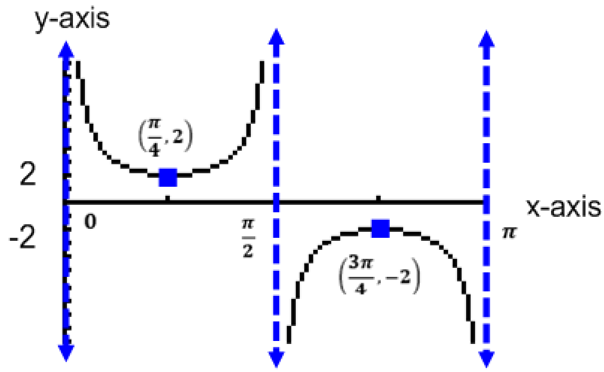
$\mathbf{Ymin = -10}$

$\mathbf{Ymax = 10}$

$\mathbf{Yscl = 1}$

Step 5: **Press**

GRAPH



Example 2: Use the graphing calculator to graph $y = 3\sec\frac{x}{2}$ from $-\pi$ to 5π.

The function $y = 3\sec\frac{x}{2}$ is in the form of $y = A\sec(Bx - C) + D$ with $A = 3$, $B = \frac{1}{2}$, $C = 0$, $D = 0$.

$$Amplitude\ is|A| = |3| = 3$$

$$Period = \frac{2\pi}{\frac{1}{2}} = 2\pi * 2 = 4\pi$$

$$Starting\ point = \frac{C}{B} = \frac{0}{2} = 0$$

$$\frac{Period}{4} = \frac{4\pi}{4} = \pi$$

The five x-values are $0, \pi, 2\pi, 3\pi$ and 4π. Since $\sec x = \frac{1}{\cos x}$, we will view $y = 3\sec\frac{x}{2}$ as

$y = \frac{3}{cos\frac{1}{2}x}$.

$$x = 0, y = \frac{3}{\cos\frac{1}{2}\cdot 0} = \frac{3}{\cos 0} = \frac{3}{1} = 3$$

$$x = \pi, y = \frac{3}{\cos\frac{1}{2}\cdot\pi} = \frac{3}{\cos\frac{\pi}{2}} = \frac{3}{0} = undefined$$

$$x = 2\pi, y = \frac{3}{\cos\frac{1}{2}\cdot 2\pi} = \frac{3}{\cos\pi} = \frac{3}{-1} = -3$$

$$x = 3\pi, y = \frac{3}{\cos\frac{1}{2}\cdot 3\pi} = \frac{3}{\cos\frac{3\pi}{2}} = \frac{3}{0} = undefined$$

$$x = 4\pi, y = \frac{3}{\cos\frac{1}{2}\cdot 4\pi} = \frac{3}{\cos 2\pi} = \frac{3}{1} = 3$$

Step 1: **Press**

Y=

Step 2: **Enter**

3 ÷ (cos (1
÷ 2) X,T,Θ,n))

Step 3: **Press**

WINDOW

Step 4: **Set the appropriate viewing window.**

$\mathbf{Xmin} = -\pi$

$\mathbf{Xmax} = 5\pi$

$\mathbf{Xscl} = \pi$

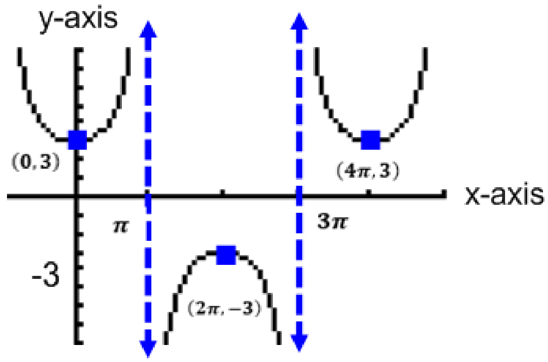
$\mathbf{Ymin} = -8$

$\mathbf{Ymax} = 8$

$\mathbf{Yscl} = 1$

Step 5: **Press**

GRAPH



2.1.4 LEARNING ACTIVITY

a. Use the graphing calculator to graph one period of $y = 3\csc 2x$.

b. Use the graphing calculator to graph one period of $y = 2\sec\frac{x}{2}$ from $-\pi$ to 5π.

2.2 Inverse Trigonometric Functions

An ***Inverse Relation*** is the interchange between the *x* and *y* variables or the *x* and *y* coordinates. For a function to have its inverse, the function must pass the horizontal line test and is a one-to-one function. A ***Horizontal Line Test*** determines a one-to-one function if a horizontal line intersects the graphs at only one point. A ***One-to-One Function*** is a function in which no two coordinates have the same *y*-value. In figure 2.2a, the sine function with domain $(-\infty, \infty)$ is not a one-to-one and has no its inverse function. If we only take a portion of the sine function $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, then the portion with restricted domain will pass the horizontal line test and is a one-to-one function (See figure 2.2b). In this section, we will begin to study how to find the acute angles, and the inverse properties of sine, cosine, and tangent.

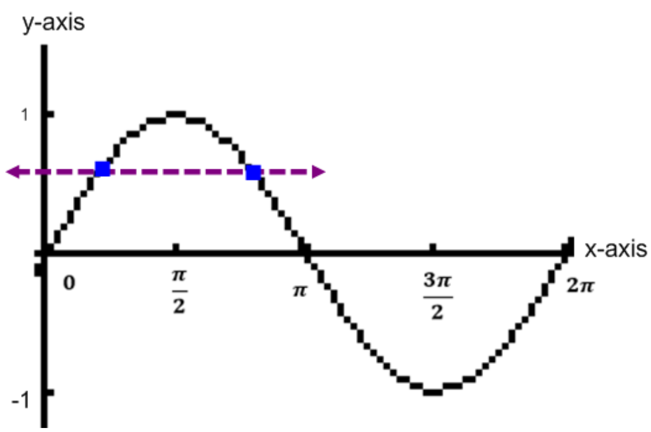


Figure 2.2a: The graph of $y = \sin x$ that does not pass horizontal line test and is not one-to-one.

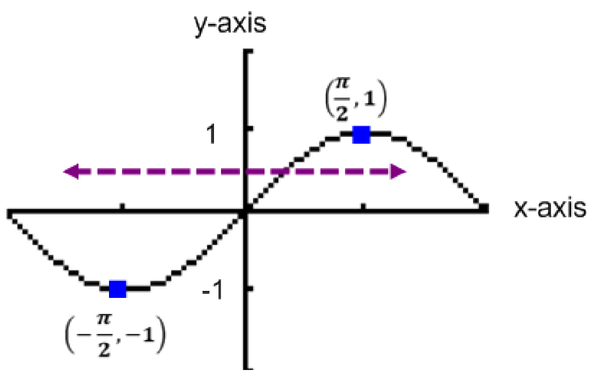


Figure 2.2b: The graph of $y = \sin x$ with a restricted domain that passes horizontal line test and is one-to-one.

2.2.1 INVERSE TRIGONOMETRIC FUNCTIONS

On the restricted domain $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, the sine function has an inverse which is called inverse sine function (See figure 2.2.1a). The two notations commonly used for the inverse sine functions are denoted as $y = \sin^{-1} x$ or $y = \arcsin x$. To find the acute angle given a trigonometric ratio, we will apply the definition of trigonometric functions, special right triangles, trigonometric function at any angles, reference angle, and the Pythagorean Theorem.

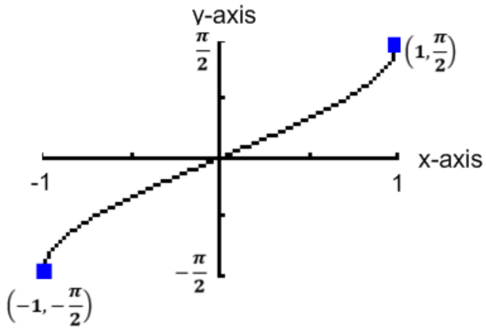


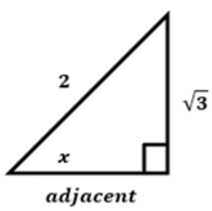
Figure 2.2.1a: The graph of $y = \sin^{-1} x$ with domain $[-1,1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAMPLE

Please work through the following example before completing the 2.2.1 LEARNING ACTIVITY:

Example: Find the exact value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

Since $\sin x = \frac{\text{opposite side of x}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$ and the ratio is positve, we can draw a right triangle in the first quadrant and find the missing side using the Pythagorean Theorem.

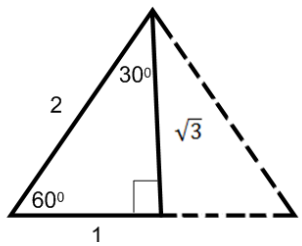


$$(adjacent)^2 = \sqrt{(2)^2 - \left(\sqrt{3}\right)^2}$$

$$= \sqrt{4-3}$$

$$= 1$$

Recall, the $30° - 60°$ special right triangle where the hypotenuse is 2 and the legs are 1 and $\sqrt{3}$. Thus, $x = 60°$.



Now, convert $60°$ to radian, we get

$$60°=60°\times\frac{\pi \text{ radians}}{180°}=\frac{\pi}{3} \quad \text{reduce } 60° \text{ and } 180° \text{ by } 60$$

So, $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$.

A cosine function with domain $(-\infty,\infty)$ does not pass the horizontal line test and is not one-to-one (See figure 2.2.1b). If we restrict the domain $0 \le x \le \pi$, then the graph of cosine function is one-to-one and has an inverse which is called inverse cosine function (See figure 2.2.1c and 2.2.1d).

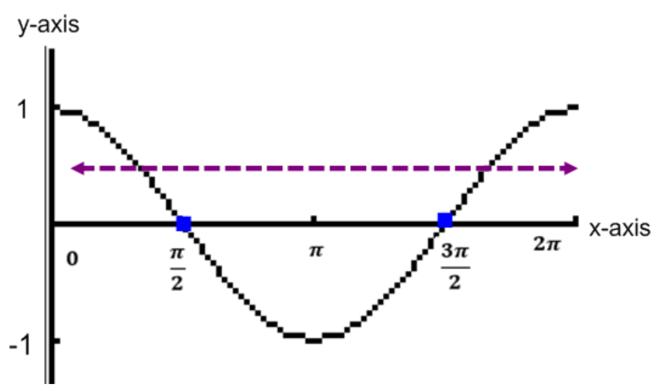


Figure 2.2.1b: The graph of $y = \cos x$ that does not pass horizontal line test and is not one-to-one.

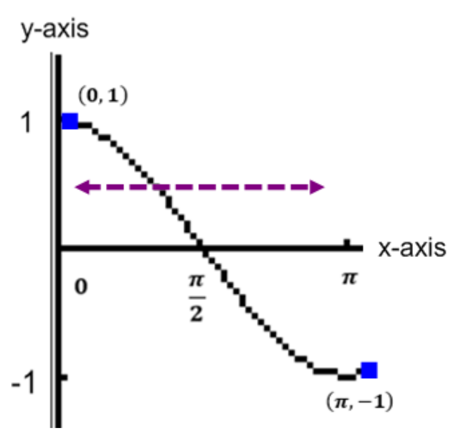


Figure 2.2.1c: The graph of $y = \sin x$ with a restricted domain that passes horizontal line test and is one-to-one.

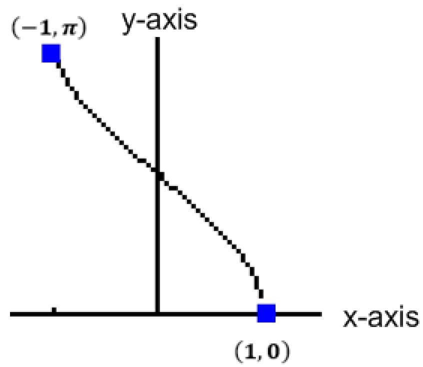


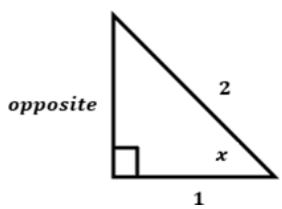
Figure 2.2.1d: The graph of $y = \cos^{-1} x$ with domain $[-1,1]$ and range $[0, \pi]$.

EXAMPLE

Please work through the following example before completing the 2.2.1 LEARNING ACTIVITY:

Example: Find the exact value of $\cos^{-1}\left(-\frac{1}{2}\right)$.

Since $\cos x = \frac{\text{adjacent side of x}}{\text{hypotenuse}} = -\frac{1}{2}$ and the ratio is negative, we can draw a right triangle in the second quadrant and find the missing side using the Pythagorean Theorem.

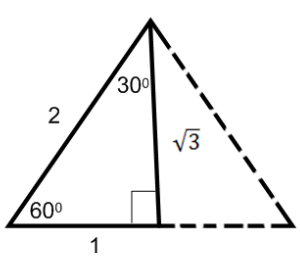


$$(\boldsymbol{opposite})^2 = \sqrt{(\mathbf{2})^2 - (-\mathbf{1})^2}$$

$$= \sqrt{\mathbf{4} - \mathbf{1}}$$

$$= \sqrt{\mathbf{3}}$$

Recall, the $30° - 60°$ special right triangle where the hypotenuse is 2 and the legs are 1 and $\sqrt{3}$. Thus, $x = 60°$.



Since the right triangle is drawn in the second quadrant, the reference angle for $60°$ is

$\mathbf{180° - \theta = 180° - 60° = 120°.}$

Now, convert $120°$ to radian, we get

$$120° = 120° \times \frac{\pi \text{ radians}}{180°} = \frac{2\pi}{3} \quad \text{reduce } 120° \text{ and } 180° \text{ by } 60$$

So, $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$.

A tangent function with domain that is all real numbers except at every odd multiple of $\frac{\pi}{2}$ does not pass the horizontal line test and is not one-to-one (See figure 2.2.1e).). If we restrict the domain $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then the graph of tangent function is one-to-one and has an inverse which is called inverse tangent function (See figure 2.2.1f and 2.2.1g).

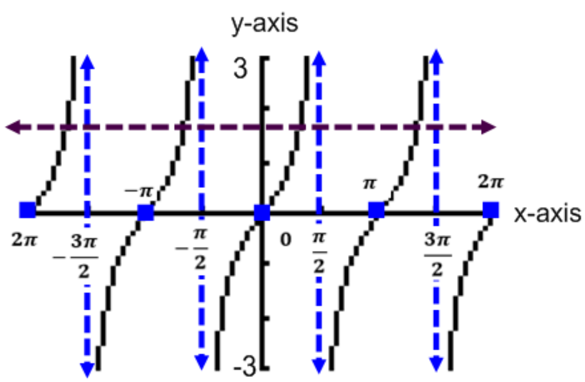


Figure 2.2.1e: The graph of $y = \tan x$ that does not pass horizontal line test and is not one-to-one.

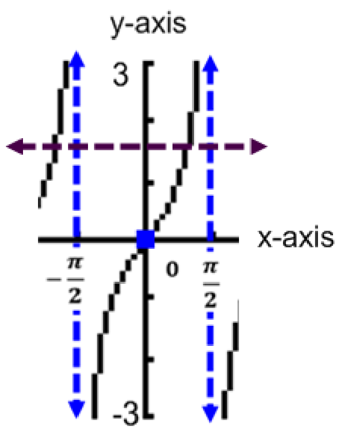


Figure 2.2.1f: The graph of $y = \tan x$ with a restricted domain that passes horizontal line test and is one-to-one.

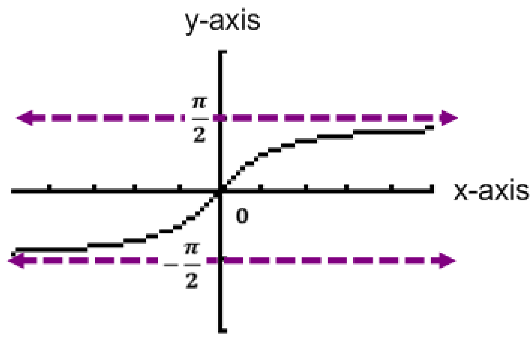


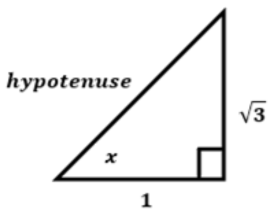
Figure 2.2.1g: The graph of $y = \tan^{-1} x$ with domain $(-\infty, \infty)$ and range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

EXAMPLE

<u>Please work through the following example before completing the 2.2.1 LEARNING ACTIVITY:</u>

Example: Find the exact value of $\tan^{-1}(\sqrt{3})$.

Since $\tan x = \frac{\text{opposite side of x}}{\text{adjacent side of x}} = \frac{\sqrt{3}}{1}$ and the ratio is positive, we can draw a right triangle in the first quadrant and find the missing side using the Pythagorean Theorem.

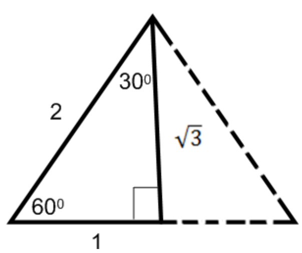


$$(\boldsymbol{hypotenuse})^2 = \sqrt{(\sqrt{3})^2 + (1)^2}$$

$$= \sqrt{3+1}$$

$$= \sqrt{4} = 2$$

Recall, the $30° - 60°$ special right triangle where the hypotenuse is 2 and the legs are 1 and $\sqrt{3}$. Thus, $x = 60°$.



Now, convert $60°$ to radian, we get

$$60°=60°\times\frac{\pi \text{ radians}}{180°}=\frac{\pi}{3} \quad \text{reduce } 60° \text{ and } 180° \text{ by } 60$$

So, $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$.

2.2.1 LEARNING ACTIVITY

a. Find the exact value of $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$.

b. Find the exact value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

c. Find the exact value of $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$.

2.2.2 Inverse Properties

The compositon of functions is where one function input into one another function, which is denoted as $(f \circ g)(x) = f\big(g(x)\big)$ where *x* is in the domain of *g* and $g(x)$ is in the domain of *f*. The compsitision of function involving inverse functon is equal to *x*, which is denoted as $f\big(f^{-1}(x)\big) = x$ and $f^{-1}\big(f(x)\big) = x$. We can apply these properties to sine, cosine, tangent, and their inverse functions to find the exact value involving composition of functon and its inverse (See table 2.2.2a).

Table 2.2.2a

Properties of Composition of Functions Involving Inverse Functions:

Compostion of Sine Function and Its Inverse:

$\sin(\sin^{-1} x) = x, \quad for\ every\ x\ in\ the\ interval\ [-1,1]$

$\sin^{-1}(\sin x) = x, \quad for\ every\ x\ in\ the\ interval\ \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Compostion of Cosine Function and Its Inverse:

$\cos(\cos^{-1} x) = x, \quad for\ every\ x\ in\ the\ interval\ [-1,1]$

$\cos^{-1}(\cos x) = x, \quad for\ every\ x\ in\ the\ interval\ [0, \pi]$

Compostion of Cosine Function and Its Inverse:

$\tan(\tan^{-1} x) = x, \quad for\ every\ x\ in\ the\ interval\ (-\infty, \infty)$

$\tan^{-1}(\tan x) = x, \quad for\ every\ x\ in\ the\ interval\ \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

EXAMPLES

Please work through the following examples before completing the 2.2.2 LEARNING ACTIVITY:

Example 1: Find the exact value of $\sin^{-1}\left(\sin \frac{\pi}{4}\right)$.

Since $x = \frac{\pi}{4}$ is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then

$\sin^{-1}\left(\sin\frac{\pi}{4}\right) = \frac{\pi}{4}$.

Example 2: Find the exact value of $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$.

Since $x = \frac{7\pi}{6}$ is not in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, then we will find $\sin\frac{7\pi}{6}$ first before we evaluate $\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$.

$\frac{7\pi}{6} \cdot \frac{180°}{\pi} = 210°$ convert to degree if it helps to locate where $\frac{7\pi}{6}$ is

So, $x = 210°$

Note that $210°$ is located in the quadrant III, thus the x-value is negative and the y-value is negative.

Now, we will begin by finding the reference angle of $210°$

$\theta' = 210° - 180° = 30°$

Since the reference angle of $\frac{7\pi}{6}$ or $210°$ is $30°$ or $\frac{\pi}{6}$ which is in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, use the $30°$-$60°$ special right triangle to find the $P = (-x, -y)$ (See figure 2.2.2a)

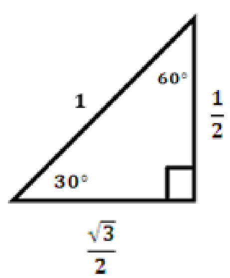


Figure 2.2.2a The hypotenuse is 1, the side opposite to $60°$ is $\frac{\sqrt{3}}{2}$, and the side opposite to $30°$ is $\frac{1}{2}$

Since $210°$ is located in the quadrant III, $\sin\frac{7\pi}{6}$ in terms of a unit circle is $\sin\frac{7\pi}{6}$ or $210° = -\frac{1}{2}$. So,

$\sin^{-1}\left(\sin\frac{7\pi}{6}\right) = \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

2.2.2 LEARNING ACTIVITY

a. Find the exact value of $\cos^{-1}\left(\cos\frac{\pi}{4}\right)$.

b. Find the exact value of $\cos^{-1}\left(\cos\frac{11\pi}{6}\right)$.

2.3 Applications of Trigonometric Functions

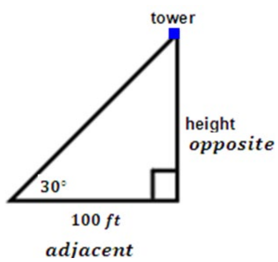
The applications of trigonometric functions involve using sine, cosine, and tangents. In the following examples, we will solve the missing side or the missing angles. To understand the problem better, sketch the condition of the problem, so it will be easier to know what we are to find.

EXAMPLES

Please work through the following examples before completing the 2.3 LEARNING ACTIVITY:

Example 1: Suppose you are standing $100\ ft$ away from a water tower with an angle of elevation of $30°$. Approximate the height of the tower to the nearest foot.

Let's draw the condition of the problem.



So, we are given an angle, an adjacent side of $30°$, and need to find the opposite side of $30°$. Therefore, we will use $\tan\theta = \frac{\text{opposite side to }\theta}{\text{adjacent side to }\theta}$.

$$\tan 30° = \frac{height}{100}$$

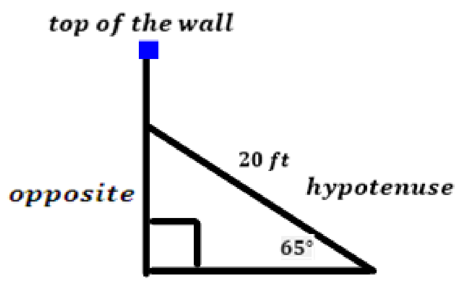
$$\frac{\tan 30°}{1} = \frac{height}{100} \quad \text{apply cross product}$$

$$height = 100 \cdot \tan 30° \quad \text{set the calculator in degree mode}$$

$$height = 58\ ft$$

Example 2: A $20\ ft$ plywood is leaning against a wall at $65°$. One end of the plywood is on the ground, and the other end is $3\ ft$ from the top of the wall. How tall is the wall to the nearest foot?

Let's draw the condition of the problem.



We will first find the opposite side of the right triangle and then add $3\ ft$ to find the height of the wall. So, we are given an angle, a hypotenuse, and need to find the opposite side of $65°$. Therefore, we will use $\sin\theta = \frac{\text{opposite side to }\theta}{\text{hypotenuse}}$.

$$\sin 65° = \frac{opposite}{20}$$

$$\frac{\sin 65°}{1} = \frac{opposite}{20} \quad \text{apply cross product}$$

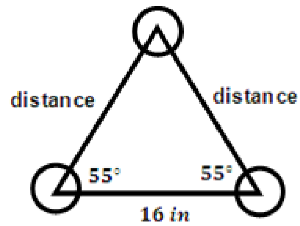
$$opposite = 20 \cdot \sin 65° \quad \text{set the calculator in degree mode}$$

$$opposite = 18\text{ ft}$$

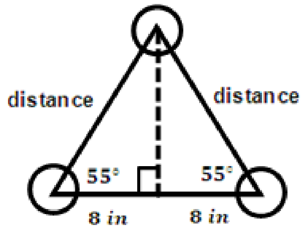
$$Wall = 18 + 3 = 21\ ft$$

Example 3: Suppose you want to drill three holes on a metal plate that will form a isoscele striangle. After drilling two holes that are $16\ in.$ apart, find the distance to the hundredth inch from either hole to the third hole if the included angle is $55°$.

Let's draw the condition of the problem.



To find the distance from either hole to the third hole, we will first bisect the isoscele triangle to create two right triangles. So, we are given an angle, an adjacent side of $55°$, and need to find the hypotenuse. Therefore, we will use $\cos\theta = \frac{\text{adjacent side to }\theta}{\text{hypotenuse}}$.



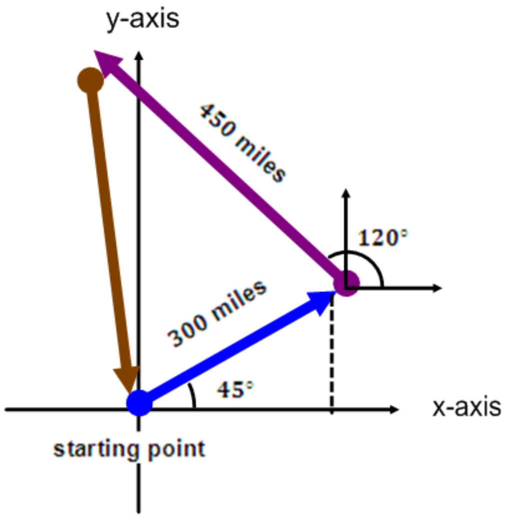
$$\cos 55° = \frac{8}{distance}$$ apply cross product

$$distance \cdot \cos 55° = 8$$ divide both sides by $\cos 55°$

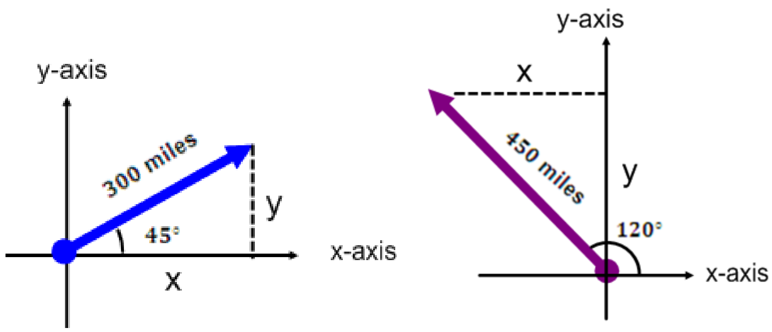
$$distance = \frac{8}{\cos 55°} = 13.95\ in$$ set the calculator in degree mode

Example 4: A vector consists of magnitude and direction. Suppose a trucker travels 300 miles at $45°$ from the starting point and then travel 450 miles at $120°$ to reach its destination. How far is the trucker from the starting point at what direction? Round the final answer to one decimal place.

We will draw the condition of the problem in the standard position.



To solve this problem, we will use the concept of component vector. We will first find the horizontal and vertical distance for both 300 miles and 450 miles and then find the total sum of each components to obtain a resultant vector, $\vec{R}$. Thus, it will tell us where the trucker stops.



For x-component, we will use $\cos\theta = \frac{\text{adjacent side to }\theta}{\text{hypotenuse}}$, and for y-component, we will use $\sin\theta = \frac{\text{opposite side to }\theta}{\text{hypotenuse}}$.

$$x - component$$

$$\cos 45° = \frac{x}{300} = 212.1320344 \quad \text{apply cross product}$$

$$\cos 120° = \frac{x}{450} = -225 \quad \text{apply cross product}$$

$$\vec{R}_x = 212.1320344 + (-225) = -21.8679656$$

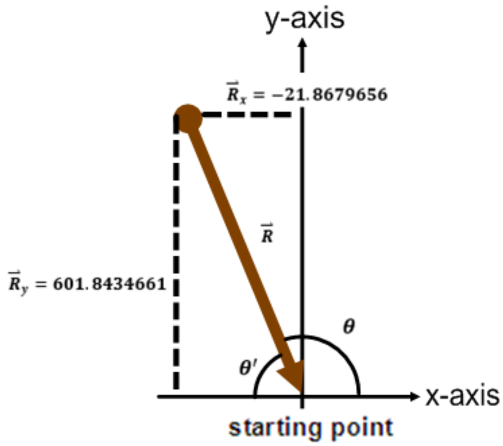
$$y - component$$

$$\sin 45° = \frac{y}{300} = 212.1320344 \quad \text{apply cross product}$$

$$\sin 120° = \frac{y}{450} = 389.7114317 \quad \text{apply cross product}$$

$$\vec{R}_y = 212.1320344 + 389.711437 = 601.8434661$$

So, the trucker is located at 21.8679656 miles to the left of the starting point and 601.8434661 miles above the origin. To find the distance from the trucker location back to the starting point, we will draw another right triangle and find the hypotenuse.



$$\vec{R} = \sqrt{\left(\vec{R}_x\right)^2 + \left(\vec{R}_y\right)^2} = \sqrt{(-21.8679656)^2 + (601.8434661)^2}$$

$\vec{R} = 602$ **miles from the starting point.**

Now, let's find the direction where the trucker's location from the starting point. Since we already found $\vec{R}_x$ and $\vec{R}_y$, we can use $\tan\theta = \frac{\text{opposite side to }\theta}{\text{adjacent side to }\theta}$ to find θ'. Then subtract θ' from $180°$ to find θ.

$$\tan\theta' = \frac{\text{opposite side to }\theta'}{\text{adjacent side to }\theta'} = \frac{601.8434661}{21.8679656}$$

$$\theta' = \tan^{-1}\left(\frac{601.8434661}{21.8679656}\right) = 87.9190758°$$

$$\theta = 180° - 87.9190758° = 92°$$

So, the trucker is 602 miles from the starting point at $92°$ angle.

2.3 LEARNING ACTIVITY

a. A news channel helicopter is flying $1250\ ft$ above the ground following a stolen car with an angle of depression $80°$. Find the distance from the helicopter to the stolen car to the nearest foot.

b. Suppose a highway ramp with 15% grade with horizontal distance of $750\ ft$. A 15% grade means that for every $100\ ft$ of horizontal distance there is a vertical rise of $15\ ft$. Find the angle of elevation to the nearest degree.

MAJOR CONCEPTS

KEY CONCEPTS

The *y*-values of sine function are the same *y*-values on the unit circle.

The *x*-values of cosine function are the same *x*-values on the unit circle.

The vertical stretching or shrinking of trigonometric functions are based on $0 < |A| < 1$ or $|A| > 1$.

$\frac{C}{B}$ determines if the trigonometric function shifts right or left.

The inverse function of sine, cosine, and tangent is graphed on the restricted domain.

Given a trigonometric ratio, we can use inverse functions to find the acute angles.

KEY TERMS

One Period/One Cycle	Maximum	Minimum	Relative Maximum
Relative Minimum	Amplitude	Inverse Relation	Horizontal Line Test
One-to-One Function			

GLOSSARY

One Period or ***One Cycle*** of a trigonometric function is the distance that a function travels on a unit circle.

The ***maximum*** is the largest y-value of a trigonometric function.

The ***minimum*** is the smallest y-value of a trigonometric function.

A ***relative maximum*** is the largest *y*-value within a certain interval of a function.

A ***relative minimum*** is the smallest *y*-value within a certain interval of a function.

An ***Amplitude***, denoted as A, of a trigonometric function is the maximum and the minimum for which the range of a trigonometric function is between $-|A|$ and $|A|$.

An ***inverse Relation*** is the interchange between the *x* and *y* variables or the *x* and *y* coordinates.

A ***Horizontal Line Test*** determines if a horizontal line intersects the graphs at more than one point.

A ***One-to-One Function*** is a function in which no two coordinates have the same *y*-value.

ASSESSMENT

ANSWERS TO LEARNING ACTIVITIES

2.1.1 LEARNING ACTIVITY

a. Identify the amplitude and the period of $y = 3\sin x$ for $0 \le x \le 2\pi$.

$$\boldsymbol{Ans\text{: } Amplitude\ is |A| = |3| = 3, Period = \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi}$$

b. Identify the amplitude and the period of $y = 2\sin\frac{1}{3}x$ for $0 \le x \le 6\pi$.

$$\boldsymbol{Ans\text{: } Amplitude\ is |A| = |2| = 2, Period = \frac{2\pi}{\frac{1}{3}} = 6\pi}$$

c. Determine whether the function is vertical shrinking, vertical stretching, horizontal shrinking, or horizontal stretching for $y = 3\sin\frac{1}{2}x$.

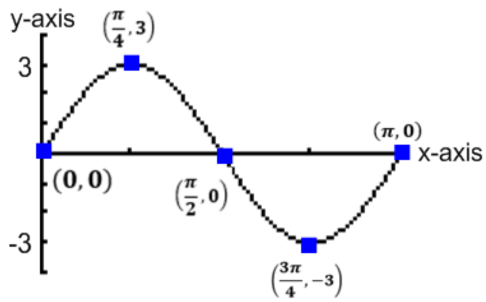
$\boldsymbol{Ans\text{: } |A| = |3| = 3 > 1}$ ***which is vertical stretching and*** $\boldsymbol{B = \frac{1}{3}}$ ***which is horizontal stretching***

d. Graph $y = 3\sin 2x$ for $0 \le x \le 2\pi$.

$$\boldsymbol{Ans\text{: } Amplitude\ is |A| = |3| = 3, \quad Period = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi, \quad \frac{Period}{4} = \frac{\pi}{4}}$$

$$\boldsymbol{x_1 = 0, y = 0, \quad x_2 = 0 + \frac{\pi}{4} = \frac{\pi}{4}, y = 3, \quad x_3 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}, y = 0,}$$

$$\boldsymbol{x_4 = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4}, y = -3, \quad x_5 = \frac{3\pi}{4} + \frac{\pi}{4} = \pi, y = 0}$$



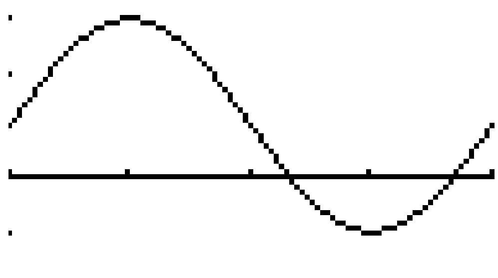
e. Identify the starting point, maximum, and minimum of $y = 2\sin\left(2x - \frac{\pi}{2}\right) + 1$.

$$\boldsymbol{Ans: Amplitude\ is\ |A| = |2| = 2, \quad Period = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi, \quad Starting\ point\ is\ \frac{C}{B} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}}$$

$$\boldsymbol{Since\ D = 1, maximum = D + |A| = 1 + |2| = 3, minimum = D - |A| = 1 - |2| = -1}$$

f. Graph $y = 2\sin\left(2x - \frac{\pi}{2}\right) + 1$ using a graphing calculator.

$$\boldsymbol{Ans: Xmin = \frac{\pi}{4}, \quad Xmax = \pi, \quad Xscl = \frac{\pi}{4}, \quad Ymin = -1, \quad Ymax = 3, \quad Yscl = 1}$$



2.1.2 LEARNING ACTIVITY

a. Identify the amplitude and the period of $y = -2\cos x$ for $0 \le x \le 2\pi$.

$$\boldsymbol{Ans: Amplitude\ is |A| = |-2| = 2, \quad Period = \frac{2\pi}{B} = \frac{2\pi}{1} = 2\pi}$$

b. Determine whether the function $y = 3\cos\frac{1}{3}x$ is vertical shrinking, vertical stretching, horizontal shrinking, or horizontal stretching.

Ans: $|A| = |3| = 3 > 1$ ***which is vertical stretching and*** $B = \frac{1}{3}$ ***which is horizontal stretching***

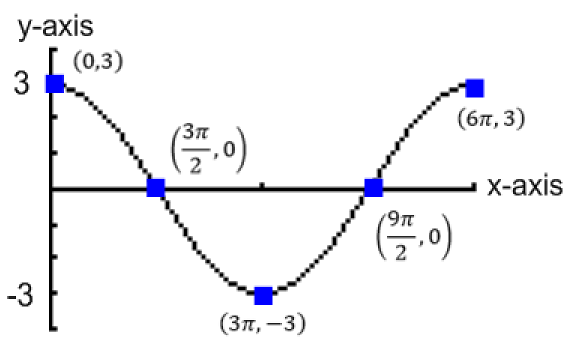
c. Graph one period of $y = 3\cos\frac{1}{3}x$.

$\boldsymbol{Ans: Amplitude\ is |A| = |3| = 3, \quad Period = \frac{2\pi}{B} = \frac{2\pi}{\frac{1}{3}} = 6\pi, \quad \frac{Period}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}}$

$\boldsymbol{Ans: Since\ D = 0, maximum = D + |A| = 0 + |3| = 3, minimum = D - |A| = 0 - |3| = -3}$

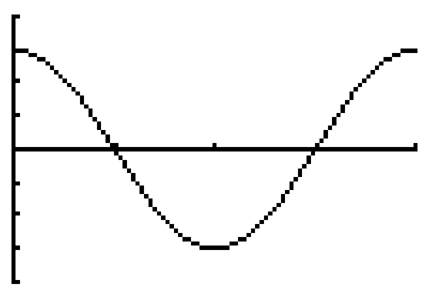
$\boldsymbol{Ans: Starting\ point\ is\ \frac{C}{B} = \frac{0}{3} = 0, \quad x_1 = 0, y = 3, \quad x_2 = 0 + \frac{3\pi}{2} = \frac{3\pi}{2}, y = 0,}$

$\boldsymbol{x_3 = \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi, y = -3, \quad x_4 = 3\pi + \frac{3\pi}{2} = \frac{9\pi}{2}, y = 0, \quad x_5 = \frac{9\pi}{2} + \frac{3\pi}{2} = 6\pi, y = 3}$



d. Use the graping calculator to graph one period of $y = 3\cos\frac{1}{3}x$.

$\boldsymbol{Ans: Xmin = 0, \quad Xmax = 6\pi, \quad Xscl = \frac{3\pi}{2}, \quad Ymin = -3, \quad Ymax = 3, \quad Yscl = 1}$



2.1.3 LEARNING ACTIVITY

a. Use the graphing calculator to graph two period of $y = \tan\frac{1}{3}x$.

$\boldsymbol{Ans: first\ pair\ of\ vertical\ asymptote:}$

$$-\frac{\pi}{2} < \frac{1}{3}x - 0 < \frac{\pi}{2}, \qquad \frac{-3\pi}{2} < x < \frac{3\pi}{2}$$

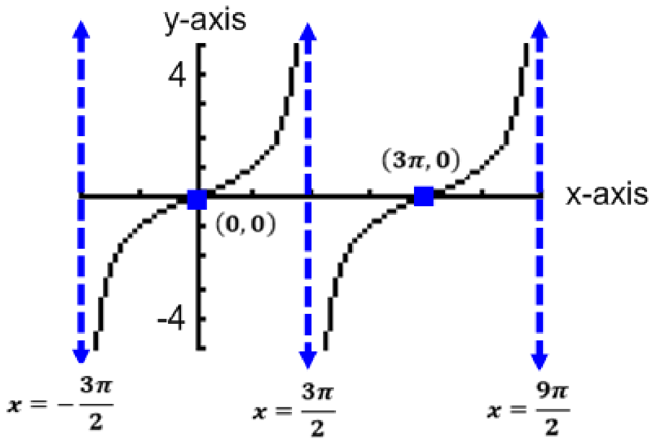
$Ans\text{: } second\ pair\ of\ vertical\ asymptote\text{:}$

$$\frac{\pi}{2} < \frac{1}{3}x - 0 < \frac{3\pi}{2}, \qquad \frac{3\pi}{2} < x < \frac{9\pi}{2}$$

Ans: The x-intercepts are midway between two consective asymptote,

$$first\ x - int. = \frac{\frac{-3\pi}{2} + \frac{3\pi}{2}}{2} = 0, \qquad second\ x - int. = \frac{\frac{3\pi}{2} + \frac{9\pi}{2}}{2} = 3\pi$$

$$Ans\text{: } Xmin = \frac{-3\pi}{2}, Xmax = \frac{9\pi}{2}, Xscl = \frac{3\pi}{4}, Ymin = -5, Ymax = 5, Yscl = 1$$



b. Use the graphing calculator to graph two period of $y = \tan\left(x - \frac{\pi}{4}\right)$.

$Ans\text{: } first\ pair\ of\ vertical\ asymptote\text{:}$

$$-\frac{\pi}{2} < x - \frac{\pi}{4} < \frac{\pi}{2}, \qquad -\frac{\pi}{4} < x < \frac{3\pi}{4}$$

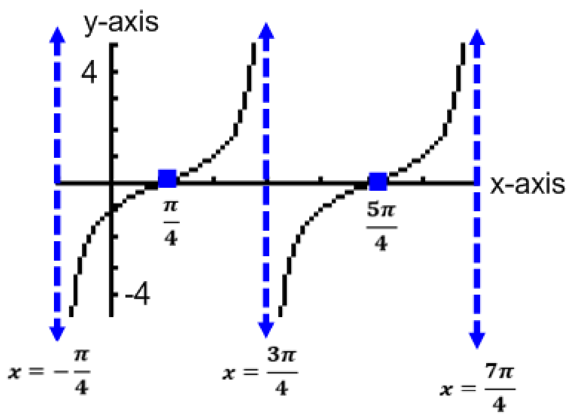
$Ans\text{: } second\ pair\ of\ vertical\ asymptote\text{:}$

$$\frac{\pi}{2} < x - \frac{\pi}{4} < \frac{3\pi}{2}, \qquad \frac{3\pi}{4} < x < \frac{7\pi}{4}$$

Ans: The x-intercepts are midway between two consective asymptote,

$$first\ x-int. = \frac{\frac{-\pi}{4}+\frac{3\pi}{4}}{2} = \frac{\pi}{4}, \qquad second\ x-int. = \frac{\frac{3\pi}{4}+\frac{7\pi}{4}}{2} = \frac{5\pi}{4}$$

$$Ans:\ Xmin = -\frac{\pi}{4}, Xmax = \frac{7\pi}{4}, Xscl = \frac{\pi}{4}, Ymin = -5, Ymax = 5, Yscl = 1$$



c. Use the graphing calculator to graph two period of $y = \cot 3x$.

$$Ans:\ first\ pair\ of\ vertical\ asymptote:$$

$$0 < 3x - 0 < \pi, \qquad 0 < x < \frac{\pi}{3}$$

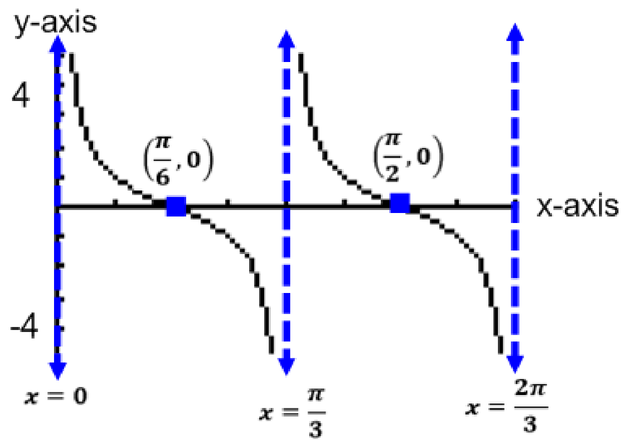
$$Ans:\ Second\ pair\ of\ vertical\ asymptote:$$

$$\pi < 3x - 0 < 2\pi, \qquad \frac{\pi}{3} < x < \frac{2\pi}{3}$$

Ans: The x-intercepts are midway between two consective asymptote,

$$First\ x-int. = \frac{0+\frac{\pi}{3}}{2} = \frac{\pi}{6}, \qquad Second\ x-int. = \frac{\frac{\pi}{3}+\frac{2\pi}{3}}{2} = \frac{\pi}{2}$$

$$Ans:\ Xmin = 0, Xmax = \frac{2\pi}{3}, Xscl = \frac{\pi}{12}, Ymin = -5, Ymax = 5, Yscl = 1$$



2.1.4 LEARNING ACTIVITY

a. Use the graping calculator to graph one period of $y = 3\csc 2x$.

Ans: $Amplitude\ is |A| = |3| = 3, Period = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

Ans: $Starting\ point = \frac{C}{B} = \frac{0}{2} = 0, \quad \frac{Period}{4} = \frac{\pi}{4}$

Ans: The five x-values are $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ *and* π. *Since* $\csc x = \frac{1}{\sin x}$, *we will view* $y = 3\csc 2x$ *as* $y = \frac{3}{\sin 2x}$.

$$x = 0, y = \frac{3}{\sin 2\cdot 0} = \frac{3}{\sin 0} = \frac{3}{0} = undefined$$

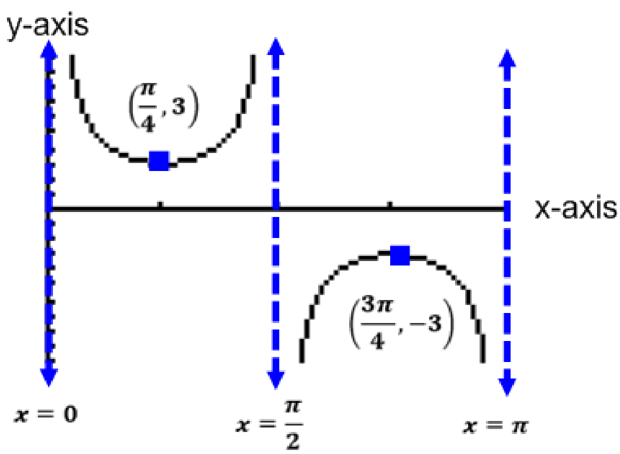
$$x = \frac{\pi}{4}, y = \frac{3}{\sin 2\cdot\frac{\pi}{4}} = \frac{3}{\sin\frac{\pi}{2}} = \frac{3}{1} = 3$$

$$x = \frac{\pi}{2}, y = \frac{3}{\sin 2\cdot\frac{\pi}{2}} = \frac{3}{\sin\pi} = \frac{3}{0} = undefined$$

$$x = \frac{3\pi}{4}, y = \frac{3}{\sin 2\cdot\frac{3\pi}{4}} = \frac{3}{\sin\frac{3\pi}{2}} = \frac{3}{-1} = -3$$

$$x = \pi, y = \frac{3}{\sin 2\cdot\pi} = \frac{3}{\sin 2\pi} = \frac{3}{0} = undefined$$

$Ans: Xmin = 0, Xmax = \pi, Xscl = \frac{\pi}{4}, Ymin = -10, Ymax = 10, Yscl = 1$



b. Use the graping calculator to graph one period of $y = 2\sec\frac{x}{2}$ from $-\pi$ to 5π.

$Ans: Amplitude\ is |A| = |2| = 2, \quad Period = \frac{2\pi}{\frac{1}{2}} = 2\pi * 2 = 4\pi$

$Ans: Starting\ point = \frac{C}{B} = \frac{0}{2} = 0, \quad \frac{Period}{4} = \frac{4\pi}{4} = \pi$

Ans: The five x-values are $0, \pi, 2\pi, 3\pi$ *and* 4π. *Since* $\sec x = \frac{1}{\cos x}$, *we will view* $y = 2\sec\frac{x}{2}$ *as*

$y = \frac{2}{\cos\frac{1}{2}x}$.

$$x = 0, y = \frac{2}{\cos\frac{1}{2}\cdot 0} = \frac{2}{\cos 0} = \frac{2}{1} = 2$$

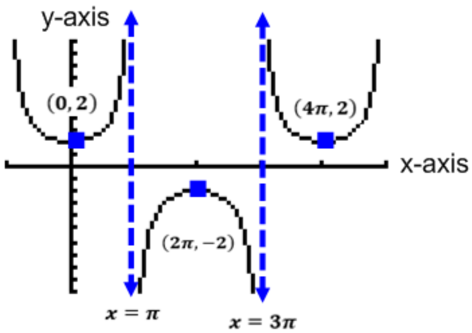
$$x = \pi, y = \frac{2}{\cos\frac{1}{2}\cdot \pi} = \frac{2}{\cos\frac{\pi}{2}} = \frac{2}{0} = undefined$$

$$x = 2\pi, y = \frac{2}{\cos\frac{1}{2}\cdot 2\pi} = \frac{2}{\cos\pi} = \frac{2}{-1} = -2$$

$$x = 3\pi, y = \frac{2}{\cos\frac{1}{2}\cdot 3\pi} = \frac{2}{\cos\frac{3\pi}{2}} = \frac{2}{0} = undefined$$

$$x = 4\pi, y = \frac{2}{\cos\frac{1}{2}\cdot 4\pi} = \frac{2}{\cos 2\pi} = \frac{2}{1} = 2$$

$$Ans\text{: } Xmin = -\pi, Xmax = 5\pi, Xscl = \pi, Ymin = -10, Ymax = 10, Yscl = 1$$



2.2.1 LEARNING ACTIVITY

a. Find the exact value of $\sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$.

$Ans\text{: } \frac{\pi}{4}$

b. Find the exact value of $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

$Ans\text{: } \frac{\pi}{6}$

c. Find the exact value of $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$.

$Ans\text{: } \frac{\pi}{6}$

2.2.2 LEARNING ACTIVITY

a. Find the exact value of $\cos^{-1}\left(\cos\frac{\pi}{4}\right)$.

$Ans\text{: } \frac{\pi}{4}$

b. Find the exact value of $\cos^{-1}\left(\cos\frac{11\pi}{6}\right)$.

$Ans\text{: } \frac{\pi}{6}$

2.3 LEARNING ACTIVITY

a. A news channel helicopter is flying $1250\ ft$ above the ground following a stolen car with an angle of depression $80°$. Find the distance from the helicopter to the stolen car to the nearest foot.

$Ans\text{: } 1269\ ft$

b. Suppose a highway ramp with 15% grade with horizontal distance of $750\ ft$. A 15% grade means that for every $100\ ft$ of horizontal distance there is a vertical rise of $15\ ft$. Find the angle of elevation to the nearest degree.
Ans: $9°$

MODULE REINFORCEMENT

SHORT ANSWER QUESTIONS

1) For the function below, determine the following:
   a) amplitude
   b) period
   c) vertical shrinking or vertical stretching
   d) horizontal shrinking or horizontal stretching

   $y = 3\sin\frac{1}{3}x$

2) Graph the following function. Be sure to state the coordinates of starting point, all the x-intercepts, maximum, and minimum of the function.
   $y = 2\sin 2x$

3) For the function below, determine the following:
   a) amplitude
   b) period
   c) starting point
   d) relative maximum
   e) relative minimum

   $y = 4\cos\left(x - \frac{\pi}{2}\right) + 1$

4) Graph one period of the following function. Be sure to state the coordinates of the five key x-values and their corresponding y-values.
   $y = 4\cos\left(x - \frac{\pi}{2}\right) + 1$

5) Graph two period of the following function. Be sure to state the vertical asymptotes and x-intercetps.
   $y = \tan\left(x - \frac{\pi}{2}\right)$

6) Graph two period of the following function. Be sure to state the vertical asymptotes and x-intercetps.
   $y = \cot\frac{1}{2}x$

7) Graph one period of the following function. Be sure to state the vertical asymptotes, relative maximum, and relative minimum.

$y = \csc\dfrac{x}{2}$

8) Graph one period of the following function. Be sure to state the vertical asymptotes, relative maximum, and relative minimum.
$y = \sec\dfrac{x}{2}$

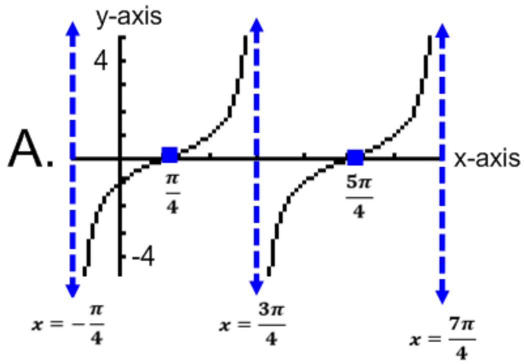
9) Find the exact value of the following inverse function.
   a) $\sin^{-1}\left(\frac{1}{2}\right)$
   b) $\cos^{-1}\left(-\frac{1}{2}\right)$
   c) $\tan^{-1}(-\sqrt{3})$

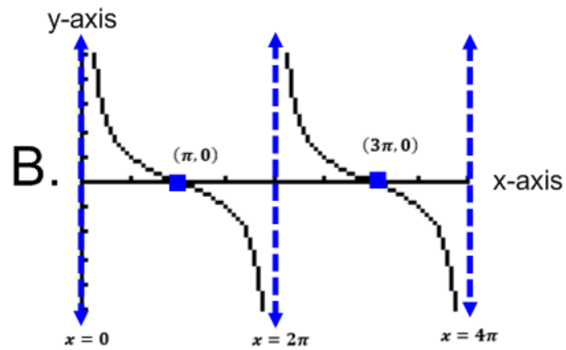
10) Find the exact value of the following inverse function.
   a) $\sin^{-1}\left(\sin\frac{\pi}{4}\right)$
   b) $\sin^{-1}\left(\sin\frac{5\pi}{4}\right)$
   c) $\tan^{-1}\left(\tan\left(-\frac{2\pi}{3}\right)\right)$

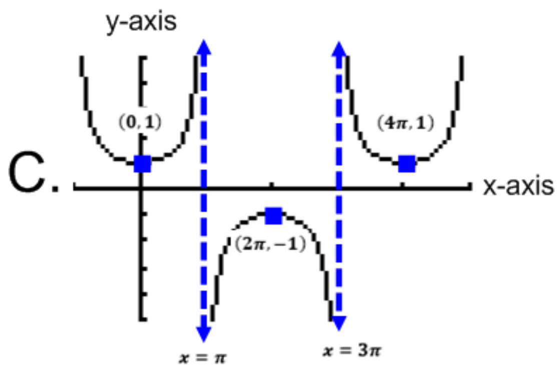
11) Car van travel 100 miles at $270°$ from the starting point and then travel 50 miles at $180°$. How far is the van from the starting point and at what direction? Round the final answer to one decimal place.

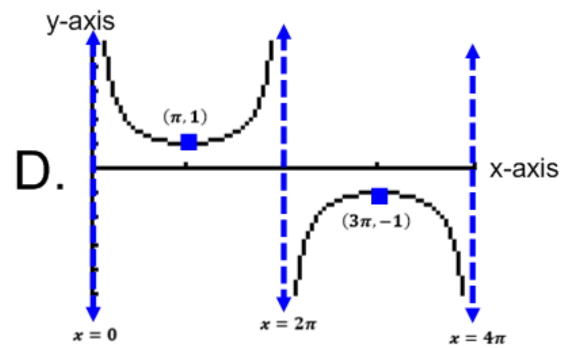
MATCHING:

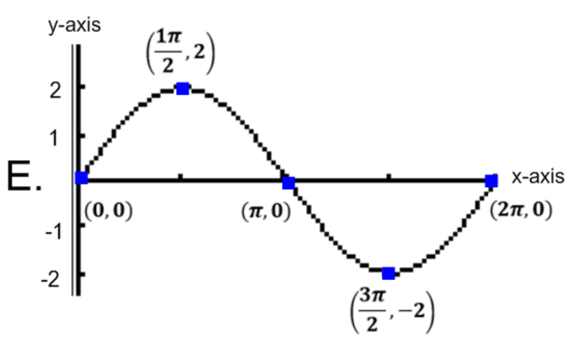
1) ___________ $y = \sec\frac{x}{2}$
2) ___________ $y = 2\sin x$
3) ___________ $y = \csc\frac{x}{2}$
4) ___________ $y = \tan\left(x - \frac{\pi}{2}\right)$
5) ___________ $y = \cot\frac{1}{2}x$
6) ___________ $y = 2\sin 2x$
7) ___________ $y = 4\cos\left(\mathrm{x} - \frac{\pi}{2}\right) + 1$
8) ___________ $y = 2\cos 3x$
9) ___________ $y = \tan\left(x - \frac{\pi}{4}\right)$
10) ___________ $y = 2\sec\frac{x}{2}$

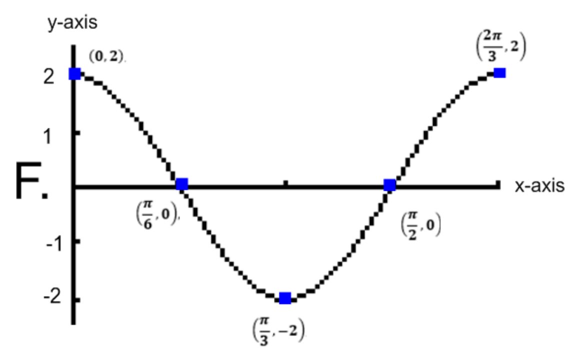

A.
y-axis
4
-4
x-axis
$\frac{\pi}{4}$
$\frac{5\pi}{4}$
$x = -\frac{\pi}{4}$
$x = \frac{3\pi}{4}$
$x = \frac{7\pi}{4}$

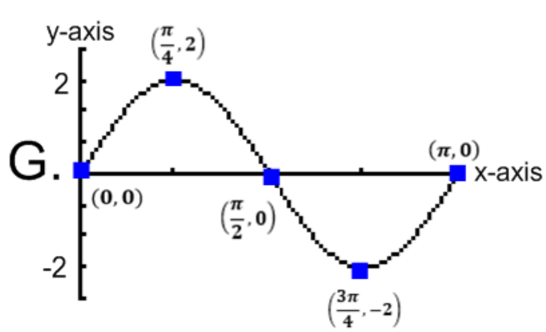

B.
y-axis
$(\pi, 0)$
$(3\pi, 0)$
x-axis
$x = 0$
$x = 2\pi$
$x = 4\pi$

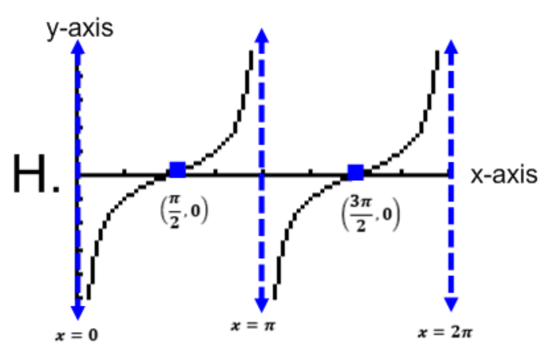

C.
y-axis
$(0, 1)$
$(4\pi, 1)$
$(2\pi, -1)$
x-axis
$x = \pi$
$x = 3\pi$


D.
y-axis
$(\pi, 1)$
$(3\pi, -1)$
x-axis
$x = 0$
$x = 2\pi$
$x = 4\pi$

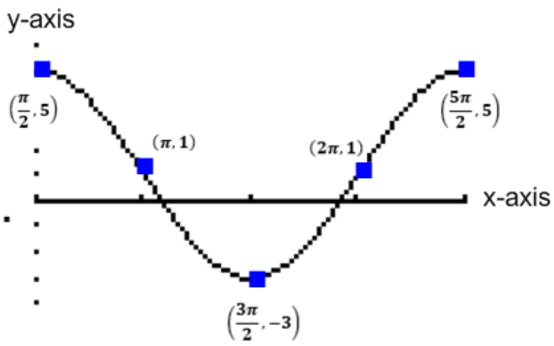

E.
y-axis
2
1
-1
-2
$\left(\frac{1\pi}{2}, 2\right)$
$(0, 0)$
$(\pi, 0)$
$(2\pi, 0)$
$\left(\frac{3\pi}{2}, -2\right)$
x-axis


F.
y-axis
2
1
-1
-2
$(0, 2)$
$\left(\frac{2\pi}{3}, 2\right)$
$\left(\frac{\pi}{6}, 0\right)$
$\left(\frac{\pi}{2}, 0\right)$
$\left(\frac{\pi}{3}, -2\right)$
x-axis

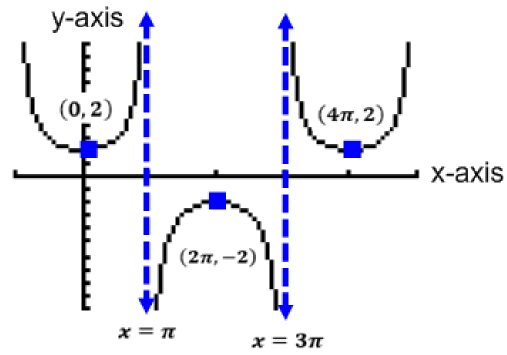

G.
y-axis
2
-2
$\left(\frac{\pi}{4}, 2\right)$
$(0, 0)$
$\left(\frac{\pi}{2}, 0\right)$
$(\pi, 0)$
$\left(\frac{3\pi}{4}, -2\right)$
x-axis


H.
y-axis
$\left(\frac{\pi}{2}, 0\right)$
$\left(\frac{3\pi}{2}, 0\right)$
x-axis
$x = 0$
$x = \pi$
$x = 2\pi$

I.


y-axis
$\left(\frac{\pi}{2}, 5\right)$
$(\pi, 1)$
$(2\pi, 1)$
$\left(\frac{5\pi}{2}, 5\right)$
x-axis
$\left(\frac{3\pi}{2}, -3\right)$

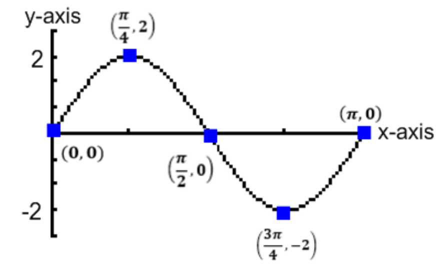
J.


y-axis
$(0, 2)$
$(4\pi, 2)$
x-axis
$(2\pi, -2)$
$x = \pi$
$x = 3\pi$

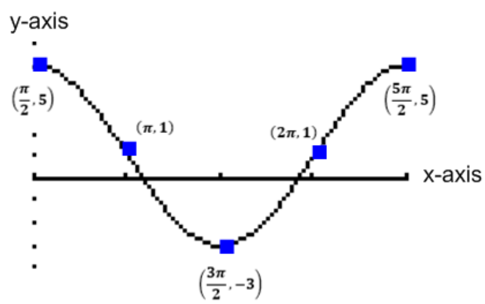
ANSWER KEY

Short Answer Questions

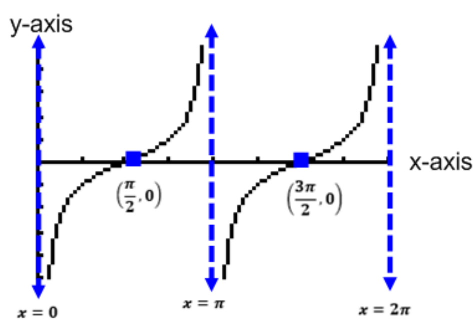
1. a) amplitude is$|A| = |3| = 3$ b) period $= \frac{2\pi}{B} = 6\pi$ c) vertical stretching
   d) horizontal stretching
2. $Starting\ Point = (0,0), x - int. \left(\frac{\pi}{2}, 0\right), (\pi, 0), Maximum = \left(\frac{\pi}{4}, 2\right), Minimum = \left(\frac{3\pi}{4}, -2\right)$



3. a) $Amplitude\ is|A| = |4| = 4$ b) $Period = \frac{2\pi}{B} = 2\pi$ c) $\frac{C}{B} = \frac{\frac{\pi}{2}}{1} = \frac{\pi}{2}$
   d) $relative\ maximum = 5$ e) $relative\ minimum = -3$
4. $x_1 = \left(\frac{\pi}{2}, 5\right), x_2 = (\pi, 1), x_3 = \left(\frac{3\pi}{2}, -3\right),\ x_4 = (2\pi, 1), x_5 = \left(\frac{5\pi}{2}, 5\right)$



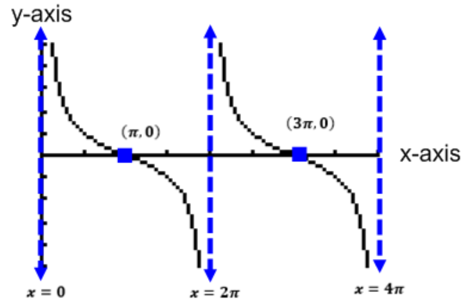
5. $Vertical\ asymptotes: x = 0, x = \pi, x = 2\pi, x - int. = \left(\frac{\pi}{2}, 0\right), \left(\frac{3\pi}{2}, 0\right)$



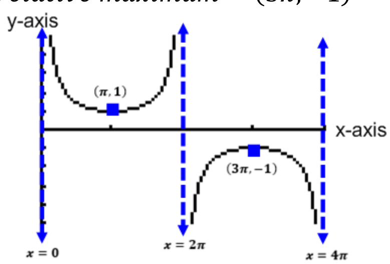
6. $Vertical\ asymptotes: x = 0, x = 2\pi, x = 4\pi,\ x - int. = (\pi, 0), (3\pi, 0)$

Matching

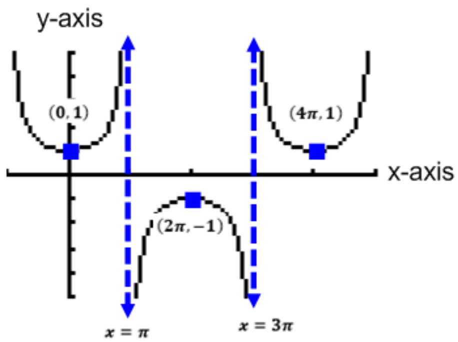
1) C
2) E
3) D
4) H
5) B
6) G
7) I
8) F
9) A
10) J



7. $Vertical\ asymptotes: x = 0, x = 2\pi, x = 4\pi relative\ minimum = (\pi, 1)$, $relative\ maximum = (3\pi, -1)$



8. $Vertical\ asymptotes: x = \pi, x = 3\pi, relative\ minimum = (0,1), (4\pi, 1)$, $relative\ maximum = (2\pi, -1)$



9. a) $\frac{\pi}{6}$ b) $\frac{2\pi}{3}$ c) $-\frac{\pi}{3}$

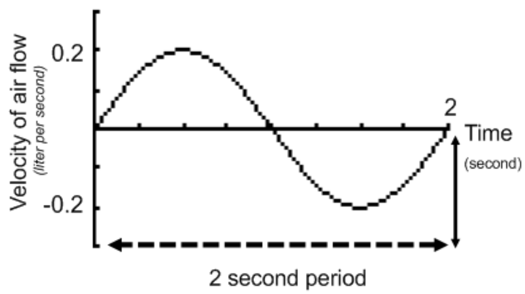
10. a) $\frac{\pi}{4}$ b) $-\frac{\pi}{4}$ c) $-\frac{2\pi}{3}$

11. $112\ miles\ at\ 243°$

CRITICAL THINKING

1) An alternating current is an example of sine function (See figure below). To find an instantaneous voltage at any angle in one period is given by $e = E_{max} \sin\theta$, where e is instantaneous voltage, E_{max} is maximum voltage, and θ is angle in degrees. Find the instantaneous voltage at $120°$ when the $E_{max} = 600\ volts$. Round the answer to the nearest tenth.
   $\boldsymbol{Ans:\ 519.6\ volts}$

2) Suppose a new born baby up to six weeks old takes at least 30 breaths (30 inhale-exhale cycles) per minutes. Suppose the inhale-exhale cycle take place every 2 seconds and the velocity of air flow measures in milliliters per second, find the function in the form $y = A\sin Bx$ that models the new born baby breathing cycle (See figure 2.1e).



   $\boldsymbol{Ans:\ 0.2\sin\pi}$

3) Describe the overall steps of graphing $y = \sin x$.
   $\boldsymbol{Ans:}$
   $\boldsymbol{Stpe\ 1:\ Find\ the\ amplitude.}$
   $\boldsymbol{Step\ 2:\ Find\ the\ period.}$
   $\boldsymbol{Step\ 3:\ Find\ starting\ point.}$
   $\boldsymbol{Step\ 4:\ \frac{Period}{4}.}$
   $\boldsymbol{Step\ 5:\ Find\ the\ five\ key\ x-values}$
   $\boldsymbol{(starting\ point, x-intercepts, maximum, and\ minimum).}$
   $\boldsymbol{Step\ 6:\ Find\ the\ corresponding\ y-values.}$
   $\boldsymbol{Step\ 7:\ Graph}$

4) The graphs of $y = \cos x$ has many similar characteristics compare to $y = \sin x$. Both graphs are called sinusoidal graphs. Examine their key similarity.
   $\boldsymbol{Ans:\ The\ graph\ of\ y = \cos x\ is\ the\ graph\ of\ y = \sin x\ with\ a\ horizontal\ shift\ of\ -\frac{\pi}{2}.}$

ATTRIBUTION TABLE

Course: MAT 111 Module 1: Trigonometric Functions

Author/s	Title	Source	License
Khan Academy	Midline, amplitude and period of a function	https://www.khanacademy.org/math/trigonometry/basic-trigonometry/trig_graphs_tutorial/v/midline-amplitude-period	CC-BY-SA-NC
Khan Academy	Midline, amplitude and period of a function	https://www.khanacademy.org/math/trigonometry/basic-trigonometry/trig_graphs_tutorial/v/midline-amplitude-period	CC-BY-SA-NC

This workforce solution was funded by a grant awarded by the U.S. Department of Labor's Employment and Training Administration. The solution was created by the grantee and does not necessarily reflect the official position of the U.S. Department of Labor. The Department of Labor makes no guarantees, warranties, or assurances of any kind, express or implied, with respect to such information, including any information on linked sites, and including, but not limited to accuracy of the information or its completeness, timeliness, usefulness, adequacy, continued availability or ownership.