3

Analytic Trigonometry

Objectives

Students will be able to:

- Define the three key terms of analytic trigonometry.
- Verify trigonometric identities using the fundamental trigonometric identities.
- Apply the sum and difference formula for cosine, sine or tangent to find the sum and difference of two angles.
- Find the exact value using the half angle formula.
- Express products between sine and cosine as sum or difference.
- Express sum or difference between sine and cosine as a product.
- Solve trigonometric equations on a certain interval.

Orienting Questions

- ✓ What are the definitions of the three key terms in this module?
- ✓ Which fundamental trigonometric identity is the appropriate identity to use when verifying trigonometric identities?
- ✓ How can the sum and difference of two angles be found?
- ✓ What is the half angle formula?
- ✓ How are products between sine and cosine as sum or difference expressed?
- ✓ How are sum or difference between sine and cosine as a product expressed?
- ✓ What concepts are used in solving trigonometric equations on a certain interval?

INTRODUCTION

In the first two modules, we examined the relationships, identities, and graphs of the six trigonometric functions. In this module, we will introduce more trigonometric identities and use them to solve trigonometric equations.

3.1 VERIFYING TRIGONOMETRIC IDENTITIES

Verifying an identity is a process where two trigonometric identities, one on each side of the equal sign, can be simplified so that they are identical to each other. There are no rigid mathematical rules on how this simplifying process should take place. Therefore, the process of verifying identities is much like solving a puzzle which requires using creative thinking and problem solving skills.

The fundamental identities such as the Reciprocal Identities, the Quotient Identities, and the Pythagorean Identities were introduced in Module One. In addition, we learned that cosine and secant functions are *even* functions and that sine, cosecant, tangent, and cotangent functions are *odd* functions (See table 3.1). An ***even function*** in which $f(-x) = f(x)$ has the graph that is symmetric to the y-axis. An ***odd function*** in which $f(-x) = -f(x)$ has the graph that is symmetric to the origin (See figure 3.1a and 3.1b).

Table 3.1

Fundamental Trigonometric Identities

Reciprocal Identities:

$$\sin x = \frac{1}{\csc x}, \quad \cos x = \frac{1}{\sec x}, \quad \tan x = \frac{1}{\cot x}, \quad \csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x}$$

Quotient Identities:

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities:

$$\sin^2\theta + \cos^2\theta = 1, \quad 1 + \tan^2\theta = \sec^2\theta, \quad 1 + \cot^2\theta = \csc^2\theta$$

Even-Odd Identities:

$$\sin(-x) = -\sin x, \quad \csc(-x) = -\csc x, \quad \tan(-x) = -\tan x, \quad \cot(-x) = -\cot x$$

$$\cos(-x) = \cos x, \quad \sec(-x) = \sec x$$

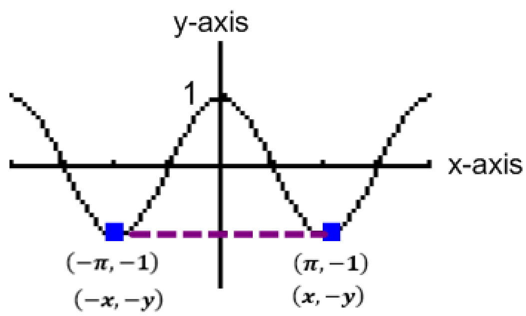


Figure 3.1a: The graph of cosine function $y = \cos x$ is an even function where $f(-x) = f(x)$ and is symmetric to the y-axis.

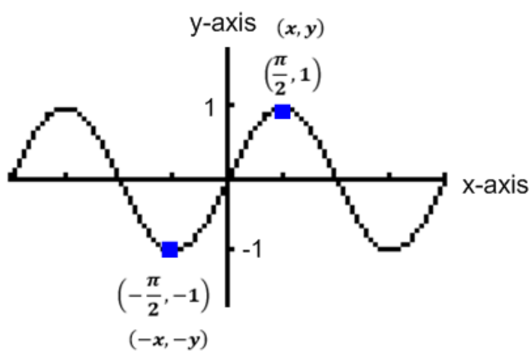


Figure 3.1b: The graph of sine function $y = \sin x$ is an odd function where $f(-x) = -f(x)$ and is symmetric to the x-axis.

Although there are no rigid ways to verify identities, we can first try to write identities in terms of sine and cosine and then look for opportunities to apply the fundamental identities.

EXAMPLES

Please work through the following examples before completing the 3.1 LEARNING ACTIVITY:

Example 1: Verify the identity:

$$\cos\theta \cdot \csc\theta = \cot\theta$$

$$\cos\theta \cdot \frac{1}{\sin\theta} = \cot\theta \quad \text{write } \csc\theta \text{ as } \frac{1}{\sin\theta}$$

$$\frac{\cos\theta}{\sin\theta} = \cot\theta$$

$$\cot\theta = \cot\theta$$

VIDEO 3.1A

Click here to watch an example of how to verify a trigonometric identity.

VIDEO 3.1A TRANSCRIPT

Video transcript of how to verify a trigonometric identity available under Module 3 in Moodle.

Example 2: Verify the identity:

$$\csc\theta \cdot \sec\theta - \tan\theta = \cot\theta$$

$$\frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta$$

$$\frac{1}{\sin\theta\cos\theta} - \frac{\sin\theta}{\cos\theta} = \cot\theta \quad \text{subtract the rational expression}$$

$$\frac{1}{\sin\theta\cos\theta} - \frac{\sin^2\theta}{\sin\theta\cos\theta} = \cot\theta$$

$$\frac{1-\sin^2\theta}{\sin\theta\cos\theta} = \cot\theta \quad \text{use } \sin^2\theta + \cos^2\theta = 1 \text{ which } \cos^2\theta = 1 - \sin^2\theta$$

$$\frac{\cos^2\theta}{\sin\theta\cos\theta} = \cot\theta \quad \text{reduce cosines}$$

$$\frac{\cos\theta}{\sin\theta} = \cot\theta$$

$$\cot\theta = \cot\theta$$

Example 3: Verify the identity:

$$\frac{1-\sin\theta}{\cos\theta} = \sec\theta - \tan\theta$$

$$\frac{1}{\cos\theta} - \frac{\sin\theta}{\cos\theta} = \sec\theta - \tan\theta \quad \text{split the numerator into two terms}$$

$$\sec\theta - \tan\theta = \sec\theta - \tan\theta \quad \textit{use } \sec x = \frac{1}{\cos x}, \textit{ and } \tan x = \frac{1}{\cot x}$$

3.1 LEARNING ACTIVITY

a. Verify the identity:
$\csc\theta \cdot \tan\theta = \sec\theta$

b. Verify the identity:
$\frac{1+\cos\theta}{\sin\theta} = \csc\theta + \cot\theta$

c. Verify the identity:
$\tan\theta - \csc\theta \cdot \sec\theta = -\cot\theta$

d. Verify the identity:
$\frac{1+\cot^2\theta}{\cos\theta} \cdot \tan\theta = \csc\theta \cdot \sec^2\theta$

3.2 SUM AND DIFFERENCE FORMULAS

Thus far, we have been using a right triangle to help us to find the trigonometric ratios of sine, cosine, and tangent for the angles that are listed on the Unit Circle. For the angles that are not listed on the Unit Circle, we can use the sum and difference formula (See table 3.2).

Table 3.2

Sum and Difference Formula:

The Sum and Difference of Two Angles for Cosine:

$$\cos(\alpha+\beta)=\cos\alpha\cdot\cos\beta-\sin\alpha\cdot\sin\beta$$

$$\cos(\alpha-\beta)=\cos\alpha\cdot\cos\beta+\sin\alpha\cdot\sin\beta$$

The Sum and Difference of Two Angles for Sine:

$$\sin(\alpha+\beta)=\sin\alpha\cdot\cos\beta+\cos\alpha\cdot\sin\beta$$

$$\sin(\alpha-\beta)=\sin\alpha\cdot\cos\beta-\cos\alpha\cdot\sin\beta$$

The Sum and Difference of Two Angles for Tangent:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\cdot\tan\beta}$$

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$$

To find the trigonometric ratio of an angle that is not listed on the Unit Circle, we will think of either the sum or the difference of two angles from the Unit Circle. For example, since we can derive the cosine of $60°$ and cosine of $45°$, cosine of $15°$ can be written as cosine of $60°-45°$. We will use this approach for the following examples. (Refer to Section 1.4 of Module 1 for review on how to derive the coordinates on the Unit Circle.)

EXAMPLES

Please work through the following examples before completing the 3.2 LEARNING ACTIVITY:

Example 1: Find the exact value of $\cos 15°$.

$$\cos 15° = \cos(60° - 45°)$$

$$= \cos 60° \cdot \cos 45° + \sin 60° \cdot \sin 45° \quad \text{Use } \cos(\alpha - \beta) \text{ formula}$$

$$= \frac{1}{2} \cdot \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}$$

$$= \frac{\sqrt{2} + \sqrt{6}}{4}$$

Example 2: Find the exact value of $\sin 15°$.

$$\sin 15° = \sin(60° - 45°)$$

$$= \sin 60° \cdot \cos 45° - \cos 60° \cdot \sin 45° \quad \text{Use } \sin(\alpha - \beta) \text{ formula}$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 3: Find the exact value of $\tan 15°$.

$$\tan 15° = \tan(60° - 45°)$$

$$= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \cdot \tan 45°} \quad \text{Use } \tan(\alpha - \beta) \text{ formula}$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3} \cdot 1} \quad \textit{Note}: \tan 60° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}, \quad \tan 45° = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$= \frac{\sqrt{3} - 1}{1 + \sqrt{3}} \times \frac{1 - \sqrt{3}}{1 - \sqrt{3}} \quad \text{Rationalize the denominator}$$

$$= \frac{\sqrt{3} - 3 - 1 + \sqrt{3}}{1 - 3} \quad \text{Apply FOIL Method for numerator and denominator}$$

$$= \frac{-4 + 2\sqrt{3}}{-2} = 2 - \sqrt{3}$$

Example 4: Find the exact value of $\cos 105°$.

$$\cos 105° = \cos(60° + 45°)$$

$$= \cos 60° \cdot \cos 45° - \sin 60° \cdot \sin 45° \quad \text{Use } \cos(\alpha+\beta) \text{ formula}$$

$$= \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4}$$

$$= \frac{\sqrt{2}-\sqrt{6}}{4}$$

Example 5: Find the exact value of $\sin 105°$.

$$\sin 105° = \sin(60° + 45°)$$

$$= \sin 60° \cdot \cos 45° + \cos 60° \cdot \sin 45° \quad \text{Use } \sin(\alpha+\beta) \text{ formula}$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

Example 6: Find the exact value of $\tan 105°$.

$$\tan 15° = \tan(60° + 45°)$$

$$= \frac{\tan 60° + \tan 45°}{1 - \tan 60° \cdot \tan 45°} \quad \text{Use } \tan(\alpha+\beta) \text{ formula}$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}\cdot 1} \quad \textit{Note}\text{: } \tan 60° = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}, \qquad \tan 45° = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

$$= \frac{\sqrt{3}+1}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}} \quad \text{Rationalize the denominator}$$

$$= \frac{\sqrt{3}+3+1+\sqrt{3}}{1-3}$$ **Apply FOIL Method for numerator and denominator**

$$= \frac{4+2\sqrt{3}}{-2} = -2-\sqrt{3}$$

VIDEO 3.2A

Click here to watch an example of how to find the exact value using the sum and difference formula.

VIDEO 3.2A TRANSCRIPT

Video transcript of how to find the exact value using the sum and difference formula available under Module 3 in Moodle.

3.2 LEARNING ACTIVITY

a. Find the exact value of $\cos(240° - 45°)$.

b. Find the exact value of $\sin(240° - 45°)$.

c. Find the exact value of $\tan(240° - 45°)$.

d. Find the exact value of $\cos(135° + 150°)$.

e. Find the exact value of $\sin(135° + 150°)$.

f. Find the exact value of $\tan(135° + 150°)$.

3.3 DOUBLE ANGLE, POWER-REDUCING, AND HALF ANGLE FORMULAS

After learning about the sum and difference of two angles of sine, cosine, and tangent, we can derive additional identities such as the double angle formula (See table 3.3a).

Table 3.3a

Double Angle Formula:

$\sin 2\theta = \sin(\theta + \theta) = 2\sin\theta \cdot \cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$

$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

For $\sin 2\theta$, we can use $\sin(\alpha + \beta) = \sin\alpha \cdot \cos\beta + \cos\alpha \cdot \sin\beta$ to help us to figure out how $\sin 2\theta$ comes about.

EXAMPLES

Please work through the following examples before completing the 3.3 LEARNING ACTIVITY:

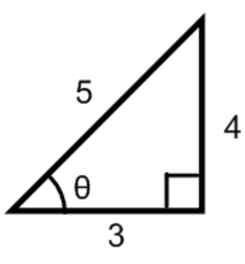
Example 1: Derive $\sin 2\theta$.

$$\boldsymbol{\sin 2\theta = \sin(\theta + \theta)} \quad \text{use } \sin(\alpha + \beta)$$

$$= \boldsymbol{\sin\theta \cdot \cos\theta + \cos\theta \cdot \sin\theta}$$

$$= \boldsymbol{2\sin\theta \cdot \cos\theta}$$

Example 2: Use the figure below to find the exact value of $\sin 2\theta$.



$$\boldsymbol{\sin 2\theta = 2\sin\theta \cdot \cos\theta}$$

$= 2\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)$ use $\sin\theta = \frac{\text{opposite side to }\theta}{\text{hypotenuse}}$ and $\cos\theta = \frac{\text{adjacent side to }\theta}{\text{hypotenuse}}$

$= \frac{24}{25}$

When sine, cosine, and tangent is squared, we can use the power-reducing formula to reduce the exponents to the first power (See table 3.3b). Power-reducing formula for sine, cosine, and tangent can be derived from $\cos 2\theta$ formula.

Table 3.3b

Power Reducing Formula:

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}, \quad \cos^2\theta = \frac{1+\cos 2\theta}{2}, \quad \tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

EXAMPLES

Please work through the following example before completing the 3.3 LEARNING ACTIVITY:

Example: Derive $\sin^2\theta$.

$\cos 2\theta = 1 - 2\sin^2\theta$ use $\cos 2\theta$ that only has sine in the formula

$2\sin^2\theta = 1 - \cos 2\theta$ divide both sides by 2

$\sin^2\theta = \frac{1-\cos 2\theta}{2}$

If we know the exact value for sine, cosine, or tangent of an angle, we can use the half angle formula to find the exact value of half that angle (See table 3.3c). For example, we know cosine of $45°$ is $\frac{1}{2}$, we can find the exact value of $\cos 22.5°$.

Table 3.3c

Half Angle Formula:

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}, \quad \cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}, \quad \tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

In table 3.3c, the $\pm$ sign represent the signs of the trigonometric function based on the quandrant that $\frac{\theta}{2}$ lies.

EXAMPLES

Please work through the following examples before completing the 3.3 LEARNING ACTIVITY:

Example 1: Find the exact value of $\cos 22.5°$.

$$\cos 22.5° = \cos\left(\frac{45°}{2}\right)$$

$$= \sqrt{\frac{1+\cos 45°}{2}} \quad \text{recall } \cos 45° = \frac{\sqrt{2}}{2}$$

$$= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}} = \sqrt{\frac{\frac{2+\sqrt{2}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

Example 2: Find the exact value of $\sin 105°$.

$$\sin 105° = \sin\frac{210°}{2}$$

$$= \sqrt{\frac{1-\cos 210°}{2}} \quad \text{recall } \cos 210° = -\frac{\sqrt{3}}{2}$$

$$= \sqrt{\frac{1-\left(-\frac{\sqrt{3}}{2}\right)}{2}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{2}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2+\sqrt{3}}}{2}$$

Example 3: Find the exact value of $\tan 75°$.

$$\tan 75° = \tan\frac{150°}{2}$$

$$= \sqrt{\frac{1 - \cos 150^\circ}{1 + \cos 150^\circ}} \quad \text{recall } \cos 150^\circ = -\frac{\sqrt{3}}{2}$$

$$= \sqrt{\frac{1 - \left(-\frac{\sqrt{3}}{2}\right)}{1 + \left(-\frac{\sqrt{3}}{2}\right)}} = \sqrt{\frac{\frac{2+\sqrt{3}}{2}}{\frac{2-\sqrt{3}}{2}}}$$

$$= \sqrt{\frac{2+\sqrt{3}}{2} \times \frac{2}{2-\sqrt{3}}} = \sqrt{\frac{2+\sqrt{3}}{2-\sqrt{3}}}$$

VIDEO 3.3A

Click here to watch an example of how to find the exact value using the half angle formula.

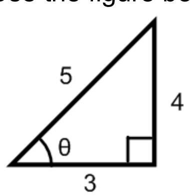
VIDEO 3.3A TRANSCRIPT

Video transcript of how to find the exact value using the half angle formula available under Module 3 in Moodle.

3.3 LEARNING ACTIVITY

a. Derive $\cos 2\theta$.

b. Use the figure below to find the exact value of $\cos 2\theta$. Assume θ lies in the first quadrant.



c. Derive $\cos^2\theta$.

d. Find the exact value of $\sin 157.5°$.

e. Find the exact value of $\cos 157.5°$.

f. Find the exact value of $\tan 157.5°$.

3.4 PRODUCT TO SUM AND SUM TO PRODUCT FORMULAS

By taking a closer look at the sum and difference of two angles of sine and cosine, we notice the formulas involve products between sine and cosine. Therefore, we can write the products between sine and cosine as the sum or difference and write the sum or difference as a product (See table 3.4a and 3.4b).

Table 3.4a

Product-to-Sum Formulas:

$$\sin\alpha \cdot \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha \cdot \cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha \cdot \sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Table 3.4b

Sum-to-Product Formulas:

$$\sin\alpha + \sin\beta = 2\cdot\sin\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cdot\sin\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cdot\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\cdot\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha+\beta}{2}$$

EXAMPLES

Please work through the following examples before completing the 3.4 LEARNING ACTIVITY:

Example 1: Express $\sin 7x \cdot \cos 3x$ as sum or difference.

$$\sin 7x \cdot \cos 3x$$

$$= \frac{1}{2}[\sin(7x+3x) + \sin(7x-4x)]$$

$$= \frac{1}{2}(\sin 10x + \sin 3x)$$

Example 2: Express $\cos 8x \cdot \sin 2x$ as sum or difference.

$$\cos 8x \cdot \sin 2x$$

$$= \frac{1}{2}[\sin(8x+2x) - \sin(8x-2x)]$$

$$= \frac{1}{2}(\sin 10x - \sin 6x)$$

Example 3: Express $\sin 3x + \sin 2x$ as a product.

$$\sin 3x + \sin 2x$$

$$= 2\sin\frac{3x+2x}{2} \cdot \cos\frac{3x-2x}{2}$$

$$= 2\sin\frac{5x}{2} \cdot \cos\frac{x}{2}$$

3.4 LEARNING ACTIVITY

a. Express $\sin 3x \cdot \sin 2x$ as sum or difference.

b. Express $\sin 3x - \sin 2x$ as a product.

3.5 TRIGONOMETRIC EQUATIONS

A ***trigonometric equation*** is an equation that contains a trigonometric expression. For example, $\sin x = \frac{1}{2}$ is a trigonometric equation where x is the variable in terms of radian. In this equation, we will find the angles of sine that are equal to $\frac{1}{2}$ in a interval of $0 \leq x < 2\pi$. If we think back to the Unit Circle in Module One, sine of $\frac{\pi}{6}$ is equal to $\frac{1}{2}$ in the first quadrant. By the definition of the trigonometric functions in terms of a Unit Circle, the $\frac{1}{2}$ is the y-value of sine, and there is another y-value that is positive in the second quadrant. Therefore, if we use the concept of reference angles from Module One, $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$ is another solution. So, $\sin x = \frac{1}{2}$ has solutions $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

If needed, we can use the special right triangles to find the angle in the first quadrant then find other solutions.

EXAMPLES

Please work through the following examples before completing the 3.5 LEARNING ACTIVITY:

Example 1: Solve $\sin x = -\frac{1}{2}, 0 < x \leq 2\pi$.

Since sine of $\frac{\pi}{6}$ is equal to $\frac{1}{2}$ in the first quadrant, we can add π to $\frac{\pi}{6}$ to find sine of x equal to $-\frac{1}{2}$ in the third quadrant.

$$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

Since of x is equal to $-\frac{1}{2}$ also in the fourth quadrant. We can subtract $\frac{\pi}{6}$ from 2π to find the other solution.

$$x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$$

So, $x = \frac{7\pi}{6}$, and $x = \frac{11\pi}{6}$.

Example 2: Solve $\tan 2x = 1, 0 < x \leq 2\pi$.

The period of tangent is π, so tangent of $\frac{\pi}{4}$ is equal to 1 in the first quadrant. Since we need to find all the solutions in the interval from 0 to 2π, any multiple of π will be our solutions.

Thus,

$$\tan 2x = 1$$

$2x = \frac{\pi}{4} + n\pi$ where n is any integer that is multiple of π

$x = \frac{\pi}{8} + \frac{n\pi}{2}$

We will start by letting n equal to 0. Once the solution exceeds 2π, we will stop.

$n = 0$

$x = \frac{\pi}{8} + \frac{0\pi}{2} = \frac{\pi}{8}$

$n = 1$

$x = \frac{\pi}{8} + \frac{1\pi}{2} = \frac{5\pi}{8}$

$n = 2$

$x = \frac{\pi}{8} + \frac{2\pi}{2} = \frac{9\pi}{8}$

$n = 3$

$x = \frac{\pi}{8} + \frac{3\pi}{2} = \frac{13\pi}{8}$

$n = 4$

$x = \frac{\pi}{8} + \frac{4\pi}{2} = \frac{17\pi}{8} > 2\pi$

So, the solutions are $x = \frac{\pi}{8}, x = \frac{5\pi}{8}, x = \frac{9\pi}{8}$, and $x = \frac{13\pi}{8}$.

Example 3: Solve $2\sin^2 x + \sin x - 1 = 0,\ 0 < x \le 2\pi$.

The equation is quadratic in form, so we will first factor the equation and then set each factor equation to zero to solve for *x*.

$2\sin^2 x + \sin x - 1 = 0$

$(2\sin x - 1)(\sin x + 1) = 0$ factoring

$2\sin x - 1 = 0,\ \sin x + 1 = 0$ set each factor equal to zero

$\sin x = \frac{1}{2},\ \ \sin x = -1$ solve for x

For $\sin x = \frac{1}{2}$,

$$x = \frac{\pi}{6} \quad \text{in the first quadrant}$$

$$x = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \quad \text{in the second quadrant}$$

For $\sin x = -1$,

$$x = \frac{3\pi}{2}$$

So, $x = \frac{\pi}{6}, x = \frac{5\pi}{6}, x = \frac{3\pi}{2}$

Example 4: Solve $2\cos^2 x + \sin x - 1 = 0,\ \ 0 < x \le 2\pi$.

Even though the equation is quadratic in form, the equation contains both sine and cosine. So we can use the trigonometric identities to write cosine as sine.

$2\cos^2 x + \sin x - 1 = 0$

$$2(1 - \sin^2 x) + \sin x - 1 = 0 \quad \text{use the identity } \cos^2\theta = 1 - \sin^2\theta$$

$$2 - 2\sin^2 x + \sin x - 1 = 0 \quad \text{distribute}$$

$$-2\sin^2 x + \sin x + 1 = 0 \quad \text{combine like terms}$$

$$2\sin^2 x - \sin x - 1 = 0 \quad \text{multiply every term by} - 1$$

$$(2\sin x + 1)(\sin x - 1) = 0 \quad \text{factoring}$$

$$2\sin x + 1 = 0, \sin x - 1 = 0 \quad \text{set each factor equal to zero}$$

$$\sin x = -\frac{1}{2}, \sin x = 1 \quad \text{solve for x}$$

For $\sin x = -\frac{1}{2}$,

$$x = \pi + \frac{\pi}{6} = \frac{7\pi}{6} \quad \text{note } \sin\frac{\pi}{6} = \frac{1}{2} \text{ and } \frac{7\pi}{6} \text{ is in the third quadrant}$$

$$x = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6} \quad \text{in the fourth quadrant}$$

For $\sin x = 1$,

$$x = \frac{\pi}{2}$$

So, $x = \frac{7\pi}{6}, x = \frac{11\pi}{6}, x = \frac{\pi}{2}$.

VIDEO 3.5A

Click here to watch an example of how to solve a trigonometric equation.

VIDEO 3.5A TRANSCRIPT

Video transcript of how to solve a trigonometric equation available under Module 3 in Moodle.

3.5 LEARNING ACTIVITY

a. Solve $\cos x = \frac{1}{2}, 0 < x \leq 2\pi$.

b. Solve $2\cos^2 x - \cos x - 1 = 0, 0 < x \leq 2\pi$.

c. Solve $\sin^2 x - 2\cos x - 1 = 0, 0 < x \leq 2\pi$.

d. Solve $\cos 2x - \sin x = 0, 0 < x \leq 2\pi$.

MAJOR CONCEPTS

KEY CONCEPTS

Write identities in terms of sine and cosine and then look for opportunities to apply the fundamental identities.

To find the trigonometric ratio of an angle that is not listed on the Unit Circle, we can use the sum or the difference of two angles from the Unit Circle.

Double angle formula can be derived from sum and difference formula.

Power-reducing formula for sine, cosine, and tangent can be derived from $\cos 2\theta$ formula.

When solving trigonometric equations, we can use the Unit Circle and the concept of reference angle.

KEY TERMS

Even Function	Odd Function	Trigonometric Equation

GLOSSARY

An ***even function*** in which $f(-x) = f(x)$ has the graph that is symmetric to the y-axis.

An ***odd function*** in which $f(-x) = -f(x)$ has the graph that is symmetric to the origin.

A ***trigonometric equation*** is an equation that contains a trigonometric expression.

ASSESSMENT

ANSWERS TO LEARNING ACTIVITIES

3.1 LEARNING ACTIVITY

a. Verify the identity:

$\csc\theta \cdot \tan\theta = \sec\theta$

Ans:

$$\frac{1}{\sin\theta} \cdot \frac{\sin\theta}{\cos\theta} = \sec\theta$$

$$\frac{1}{\cos\theta} = \sec\theta$$

b. Verify the identity:

$$\frac{1+\cos\theta}{\sin\theta} = \csc\theta + \cot\theta$$

Ans:

$$\frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta} = \csc\theta + \cot\theta$$

$$\csc\theta + \cot\theta = \csc\theta + \cot\theta$$

c. Verify the identity:

$\tan\theta - \csc\theta \cdot \sec\theta = -\cot\theta$

Ans:

$$\frac{\sin\theta}{\cos\theta} - \frac{1}{\sin\theta} \cdot \frac{1}{\cos\theta} = -\cot\theta$$

$$\frac{\sin^2\theta - 1}{\sin\theta\cos\theta} = -\cot\theta$$

$$\frac{-\cos^2\theta}{\sin\theta\cos\theta} = -\cot\theta$$

$$-\frac{\cos\theta}{\sin\theta} = -\cot\theta$$

$$-\cot\theta = -\cot\theta$$

d. Verify the identity:

$$\frac{1+\cot^2\theta}{\cos\theta} \cdot \tan\theta = \csc\theta \cdot \sec^2\theta$$

Ans:

$$\frac{csc^2\theta}{\cos\theta} \cdot \tan\theta = \csc\theta \cdot \sec^2\theta$$

$$\frac{\left(\frac{1}{\sin\theta}\right)^2}{\cos\theta} \cdot \frac{\sin\theta}{\cos\theta} = \csc\theta \cdot \sec^2\theta$$

$$\frac{\frac{1}{\sin\theta}}{\cos^2\theta} = \csc\theta \cdot \sec^2\theta$$
$$\frac{1}{\sin\theta} \cdot \frac{1}{\cos^2\theta} = \csc\theta \cdot \sec^2\theta$$
$$\csc\theta \cdot \sec^2\theta = \csc\theta \cdot \sec^2\theta$$

3.2 LEARNING ACTIVITY

a. Find the exact value of $\cos(240° - 45°)$.

$Ans: \dfrac{-\sqrt{2} - \sqrt{6}}{4}$

b. Find the exact value of $\sin(240° - 45°)$.

$Ans: \dfrac{-\sqrt{6} + \sqrt{2}}{4}$

c. Find the exact value of $\tan(240° - 45°)$.

$Ans: \dfrac{\sqrt{3} - 1}{1 + \sqrt{3}}$

d. Find the exact value of $\cos(135° + 150°)$.

$Ans: \dfrac{\sqrt{6} - \sqrt{2}}{4}$

e. Find the exact value of $\sin(135° + 150°)$.

$Ans: \dfrac{-\sqrt{6} - \sqrt{2}}{4}$

f. Find the exact value of $\tan(135° + 150°)$.

$Ans: \dfrac{-3 - \sqrt{3}}{3 - \sqrt{3}}$

3.3 LEARNING ACTIVITY

a. Derive $\cos 2\theta$.

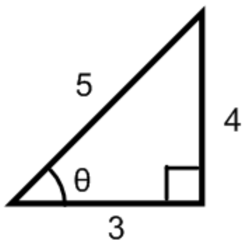
$Ans:$

$\cos 2\theta = \cos(\theta + \theta)$

$= \cos\theta \cdot \cos\theta - \sin\theta \cdot \sin\theta$

$= \cos^2\theta - \sin^2\theta$

b. Use the figure below to find the exact value of $\cos 2\theta$. Assume θ lies in the first quadrant.



Ans: $\frac{7}{25}$

c. Derive $\cos^2\theta$.

Ans:

$\cos 2\theta = 2\cos^2\theta - 1$

$2\cos^2\theta = 1 + \cos 2\theta$

$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$

d. Find the exact value of $\sin 157.5°$.

Ans: $\frac{\sqrt{2 - \sqrt{2}}}{2}$

e. Find the exact value of $\cos 157.5°$.

Ans: $-\frac{\sqrt{2 + \sqrt{2}}}{2}$

f. Find the exact value of $\tan 157.5°$.

Ans: $-\sqrt{\frac{2 - \sqrt{2}}{2 + \sqrt{2}}}$

3.4 LEARNING ACTIVITY

a. Express $\sin 3x \cdot \sin 2x$ as sum or difference.

Ans: $\frac{1}{2}(\cos x - \cos 5x)$

b. Express $\sin 3x - \sin 2x$ as a product.

Ans: $2\sin\frac{x}{2} \cdot \cos\frac{5x}{2}$

3.5 LEARNING ACTIVITY

a. Solve $\cos x = \frac{1}{2}, 0 < x \leq 2\pi$.

Ans: $x = \frac{\pi}{3}, x = \frac{5\pi}{3}$

b. Solve $2\cos^2 x - \cos x - 1 = 0, 0 < x \leq 2\pi$.

$\boldsymbol{Ans: \ x = \frac{2\pi}{3}, x = \frac{4\pi}{3}, x = 2\pi}$

c. Solve $\sin^2 x - 2\cos x = 1, 0 < x \leq 2\pi$.

$\boldsymbol{Ans: \ x = \pi}$

d. Solve $\cos 2x - \sin x = 0, 0 < x \leq 2\pi$.

$\boldsymbol{Ans: \ x = \frac{\pi}{6}, x = \frac{3\pi}{2}, x = \frac{5\pi}{6}}$

MODULE REINFORCEMENT

SHORT ANSWER QUESTIONS

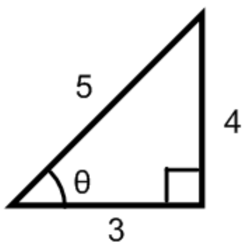
1) Verify the identity:
$\sec\theta - \sec\theta \cdot \sin^2\theta = \cos\theta$

2) Find the exact value of $\cos(240° + 45°)$.

3) Find the exact value of $\tan(240° + 45°)$.

4) Derive $\tan 2\theta$.

5) Use the figure below to find the exact value of $\tan 2\theta$. Assume θ lies in the first quadrant.



6) Find the exact value of $\cos 105°$.

7) Express $\cos 4x + \cos 2x$ as a product.

8) Solve $\cos x = -\frac{1}{2}, 0 < x \le 2\pi$.

9) Solve $2\cos^2 x + \cos x - 1 = 0, 0 < x \le 2\pi$.

10) Solve $2\cos^2 x + 3\sin x = 0, 0 < x \le 2\pi$.

MULTIPLE CHOICE: READ THE FOLLOWING QUESTIONS OR STATEMENTS AND SELECT THE BEST ANSWER.

1) __________ Is $\sin\theta \cdot \tan\theta + \cos\theta = \csc\theta$?

a. Yes
b. No

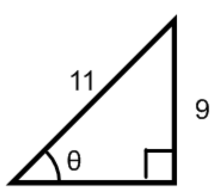
2) __________ Find the exact value of $\cos(330° - 225°)$.

a. $\frac{-\sqrt{2}+\sqrt{6}}{4}$
b. $\frac{-\sqrt{2}-\sqrt{6}}{4}$
c. $\frac{\sqrt{2}+\sqrt{6}}{4}$
d. $\frac{\sqrt{2}-\sqrt{6}}{4}$

3) __________ Find the exact value of $\tan(330° - 225°)$.

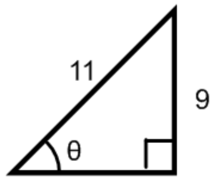
a. $\frac{-\sqrt{3}-3}{3-\sqrt{3}}$
b. $\frac{\sqrt{3}-3}{3+\sqrt{3}}$
c. $\frac{\sqrt{3}+3}{3-\sqrt{3}}$
d. $\frac{-\sqrt{3}+3}{-3-\sqrt{3}}$

4) __________ Use the figure below to find the exact value of $\sin 2\theta$. Assume θ lies in the first quadrant.



a. $\frac{18}{11}$
b. $\frac{9}{11}$
c. $\frac{18\sqrt{40}}{121}$
d. $\frac{36\sqrt{10}}{121}$

5) __________ Use the figure below to find the exact value of $\cos 2\theta$. Assume θ lies in the first quadrant.



a. $\frac{359}{121}$
b. $-\frac{41}{121}$
c. $-\frac{61}{121}$

6) __________ Find the exact value of $\tan 112.5$.

a. $-1-\sqrt{2}$
b. $1-\sqrt{2}$
c. $1+\sqrt{2}$
d. $-1+\sqrt{2}$

7) __________ Express $\sin 4x - \sin x$ as a product.

a. $2\sin\frac{5x}{2}\cdot\cos\frac{3x}{2}$
b. $2\sin\frac{3x}{2}\cdot\cos\frac{5x}{2}$
c. $2\cos\frac{5x}{2}\cdot\sin\frac{3x}{2}$
d. $2\cos\frac{3x}{2}\cdot\cos\frac{5x}{2}$

8) __________ Express $\sin 3x \cdot \sin 2x$ as sum or difference.

a. $\frac{1}{2}\sin 6x$
b. $\frac{1}{2}(\sin 5x + \sin x)$
c. $\frac{1}{2}\sin 4x$
d. $\frac{1}{2}(\cos x - \cos 5x)$

9) __________ Solve $\cos x = -\frac{\sqrt{2}}{2}, 0 < x \leq 2\pi$.

a. $x = \frac{3\pi}{4}, x = \frac{5\pi}{4}$

b. $x = \frac{\pi}{4}, x = \frac{7\pi}{4}$

10) __________ Solve $2\sin^2 x - \sin x - 3 = 0, 0 < x \leq 2\pi.$

a. $x = \frac{\pi}{2}$
b. $x = \pi$
c. $x = \frac{3\pi}{2}$

ANSWER KEY

Short Answer Questions	Multiple Choice
1) $\frac{1}{\cos\theta} - \frac{1}{\cos\theta}\cdot(1-cos^2\theta) = \cos\theta$ $\frac{1}{\cos\theta} - \frac{(1-\cos^2\theta)}{\cos\theta} = \cos\theta$ $\frac{1-1+\cos^2\theta}{\cos\theta} = \cos\theta$ $\frac{\cos^2\theta}{\cos\theta} = \cos\theta$ $\cos\theta = \cos\theta$ 2) $\frac{-\sqrt{2}+\sqrt{6}}{4}$ 3) $\frac{\sqrt{3}+1}{1-\sqrt{3}}$ 4) $\tan 2\theta = \tan(\theta+\theta)$ $= \frac{\tan\theta + \tan\theta}{1-\tan\theta\cdot\tan\theta}$ $= \frac{2\tan\theta}{1-\tan^2\theta}$ 5) $\frac{24}{7}$ 6) $-\frac{\sqrt{2}-\sqrt{6}}{4}$ 7) $2\cos 3x \cdot \cos x$ 8) $x = \frac{2\pi}{3}, x = \frac{4\pi}{3}$ 9) $x=\frac{\pi}{3}, x = \frac{5\pi}{3}, x = \pi$ 10) $x=\frac{7\pi}{6}, x = \frac{11\pi}{6}$	1) b 2) d 3) a 4) d 5) b 6) a 7) b 8) d 9) a 10) c

CRITICAL THINKING

1) Derive $\tan^2\theta$.

   $Ans: \tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$

   $$= \frac{\frac{1-\cos 2\theta}{2}}{\frac{1+\cos 2\theta}{2}} = \frac{1-\cos 2\theta}{2} \times \frac{2}{1+\cos 2\theta}$$

   $$= \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

2) For the trigonometric equation $\sin x \cdot \cos x = \frac{1}{2}, 0 \leq x < 2\pi$, which trigonometric identity will you anticipate of using?
   $Ans: \ We\ will\ mutipliy\ both\ sides\ by\ 2\ and\ anticipate\ of\ using\ trigonometric\ identity\ \sin 2x = 2\sin x \cdot \cos x.$

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