1 Trigonometric Functions

Objectives

Students will be able to:

- Define the seven key terms in trigonometric functions.
- Convert between degree and radian.
- Choose the appropriate trigonometric function to find the missing sides of a right triangle.
- Compute an acute angle by using the Pythagorean Identity.
- Use Reciprocal Identities, Quotient Identities, and Pythagorean Identities to simplify an expression.
- Calculate the reference angle.
- Calculate a coterminal angle.
- Use the graphing calculator to compute trigonometric function values.
- Use the graphing calculator to find the acute angle.
- Find the exact value of the six trigonometric functions for a given coordinate and trigonometric ratio.
- Derive the coordinate of a unit circle for the angles commonly seen in trigonometry using the concept of special right triangle, trigonometric functions, and reference angles.

Orienting Questions

- ✓ What are the definitions of the seven key terms in this module?
- ✓ What is the process to convert between degree and radian?
- ✓ How is a coterminal angle calculated?
- ✓ What are the six trigonometric functions?
- ✓ How is the missing side of a right triangle determined using trigonometric functions?
- ✓ How are the exact values of sine for a given coordinate at the standard position determined?
- ✓ How is an acute angle computed using the Pythagorean Identity?
- ✓ How are reciprocal identities, quotient identities, and Pythagorean identities used to simplify an expression?
- ✓ How are trigonometric function values and acute angles computed using a graphing calculator?
- ✓ How is the exact value of the six trigonometric functions calculated for a given coordinate and a trigonometric ratio?
- ✓ How is a reference angle located?
- ✓ How are coordinates of a unit circle derived?

INTRODUCTION

The word "trigonometry" means "measurement of triangles." Trigonometry is used every day by millions of people even though most people think it has few real-life applications. It is a type of math that is necessary in areas such as manufacturing, navigation, building construction, and even in the medical field. Basically, trigonometry is used anywhere precise distance measurement is required. For example, trigonometry is used to measure the height of things in nature, to create your favorite music, to determine the best lighting arrangement in your home and in medical procedures that diagnose health problems.

The goal of this module is to introduce the concepts of angle and radian measure, right triangles, trigonometric identities, trigonometric functions of any angles, and the unit circle. These topics will lay a solid foundation for the understanding of trigonometry.

1.1 RADIANS AND ANGLES IN STANDARD POSITION

To better understand trigonometry, we will first review angles. An ***Angle*** is formed by two rays that have a common endpoint. One ray is called the 'initial side' and the other ray is called the 'terminal side'. The common endpoint is called the 'vertex' (See Figure 1.1a).

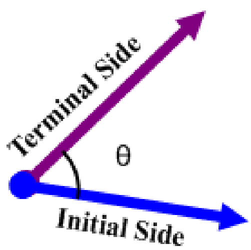


Figure 1.1a: An angle is formed by two rays. One ray is called the initial side and another ray is called the terminal side.

In trigonometry, angles are both measured in degree notation or in radian. When representing angles using variables, the angle is often labeled by Greek letters like θ (theta), α (alpha), or β (beta). Therefore, a notation such as $\theta=360°$ will be referred to as angle θ whose measure is $360°$.

So, what is a radian? Let's first define it and then we will see how it relates to degree notation.

1.1.1 RADIANS

1 Radian is the measure of a central angle of a circle that intercepts an arc where the arc length is the same as the radius. If we draw two radiuses and let them intercept the circle where the intercept arc length is also r, then the angle (central angle) that the two radiuses and the intercepted arc created is 1 radian (See Figure 1.1.1a). Thus, we can find any radian measure by taking the length of the intercepted arc divided by the circle's radius.

$$\text{radian measure}=\frac{\text{length of the intercepted arc}}{\text{radius}}$$

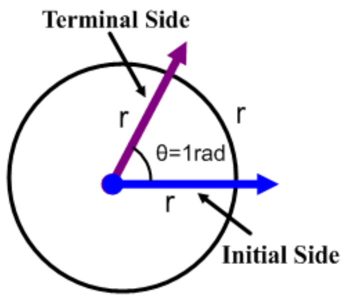


Figure 1.1.1a: For 1 radian angle, created by r unit of radius and r unit of arc length.

When the intercepted arc length doubles the radius, it will equal 2 radians (See Figure 1.1.1b).

$$\frac{\text{2r unit of arc length}}{\text{r unit of radius}} = 2 \text{ radian}$$

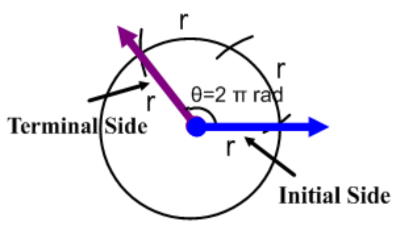


Figure 1.1.1b: For 2 radian angle, created by r unit of radius and 2r unit of arc length.

If the terminal side makes one rotation, which means the intercepted arc length is equal to the circumference of the circle, then it will equal $2\pi \text{ radian} = 360°$ (See Figure 1.1.1c).

$$\frac{\text{circumference (arc length)}}{\text{r unit of radius}} = \frac{2\pi r}{r} = 2\pi \text{ radian} = 360°$$

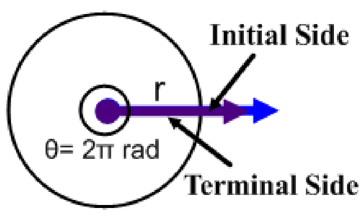


Figure 1.1.1c: Terminal side makes 1 rotation. Thus, $2\pi \text{ radian} = 360°$

Since $2\pi \text{ radian} = 360°$, the relationship between degree notation and radian measure is that they are one in the same. They are just different types of measurements. We can acutally convert between the degree and radian (See Table 1.1.1a).

Table 1.1.1a

Conversion between Degree and Radians:

If 2π radian $= 360°$, then π radians $= 180°$

To convert degree to radian: $\times \dfrac{\pi \text{ radians}}{180°}$

To convert radian to degree: $\times \dfrac{180°}{\pi \text{ radians}}$

Since π is an irrational number, radians are expressed as a fraction using multiples of π. It is not necessary to use decimal approximations.

EXAMPLES

Please work through the following examples before completing the 1.1.1 LEARNING ACTIVITY:

Example 1: Convert 18° to radians. Express the answer as a multiple of π.

$$18° = 18° \times \frac{\pi \text{ radians}}{180°} \quad \text{reduce } 18° \text{ and } 180° \text{ by } 18$$

$$= 1 \times \frac{\pi \text{ radians}}{10} = \frac{1\pi}{10} \text{ radians}$$

Example 2: Convert -270° to radians. Express the answer as a multiple of π.

$$-270° = -270° \times \frac{\pi \text{ radians}}{180°} \quad \text{reduce } 270° \text{ and } 180° \text{ by } 90$$

$$= -3 \times \frac{\pi \text{ radians}}{2} = -\frac{3\pi}{2} \text{ radians}$$

Example 3: Convert $\frac{\pi}{9}$ radians to degrees.

$$\frac{\pi}{9} \text{ radians} = \frac{\pi \text{ radians}}{9} \times \frac{180°}{\pi \text{ radians}} \quad \text{reduce } \pi \text{ radians}$$

$$= \frac{180°}{9} \quad \text{reduce } 180° \text{ and } 9 \text{ by } 9$$

$=20°$

VIDEO 1.1.1A

Click here to watch radian and degree conversion.

1.1.1 LEARNING ACTIVITY

a. Convert 330° to radians. Express the final answer as a multiple of π.

b. Convert $\frac{3\pi}{4}$ radians to degrees.

1.1.2 ANGLES IN STANDARD POSITION

After learning about how to convert between degree and radian, we will now take a look at angles in standard position. An angle is in standard position if the vertex is at the origin of a rectangular coordinate system, and the initial side of the angle lies along the positive x-axis. As the terminal side of the angle rotates counterclockwise, the angle θ is positive (See figure 1.1.2a). If the terminal side rotates clockwise, then the angle θ is negative (See figure 1.1.2b).

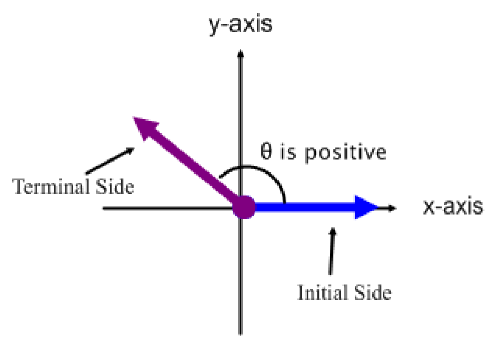


Figure 1.1.2a: Terminal side rotates counterclockwise. θ is positive.

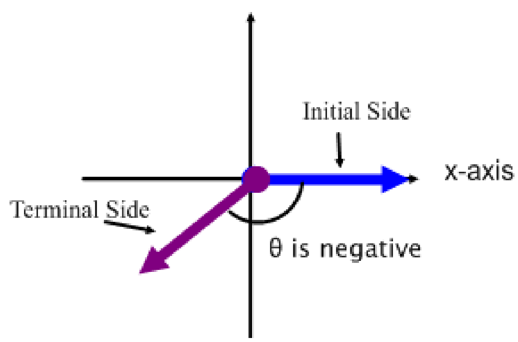


Figure 1.1.2b: Terminal side rotates clockwise. θ is negative.

Now, let us get familiar with angles in standard position. Figures 1.1.2c and 1.1.2d, illustrate both degree and radian measure of angles that are commonly seen in trigonometry. There are five basic angles we need to know first to get a sense of where other angles are located in the standard position: $0°$ is where both initial and terminal sides lie on top of one another on the x-axis; $90°$ is where the terminal side rotates counterclockwise to the y-axis above the origin; $180°$ is where the initial and terminal sides forms a straight line on the x-axis; $270°$ is where the terminal side continues to rotate counterclockwise onto the y-axis below the origin; and last, $360°$ is where the terminal side makes one rotation back to the x-axis. Since these five angles are in the standard position, we can also tell which quadrant the angles are located in since the first quadrant is from $0°$ to $90°$, the second quadrant is from $90°$ to $180°$, the third quadrant is from $180°$ to $270°$ and the fourth quadrant is from $270°$ to $360°$. Knowing which quadrant the angle is in helps us to determine the signs of the coordinates of every angle in the standard position.

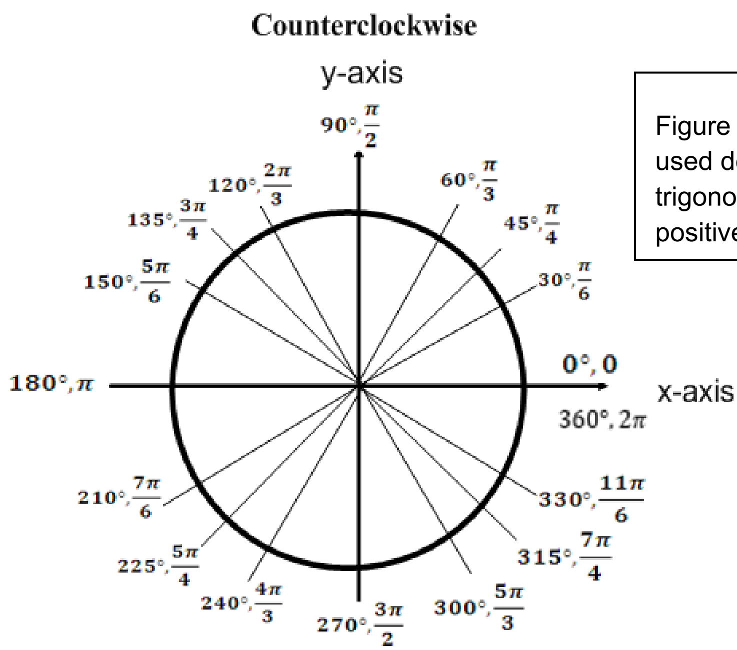


Figure 1.1.2c: Commonly used degree and radian in trigonometry where θ is positive.

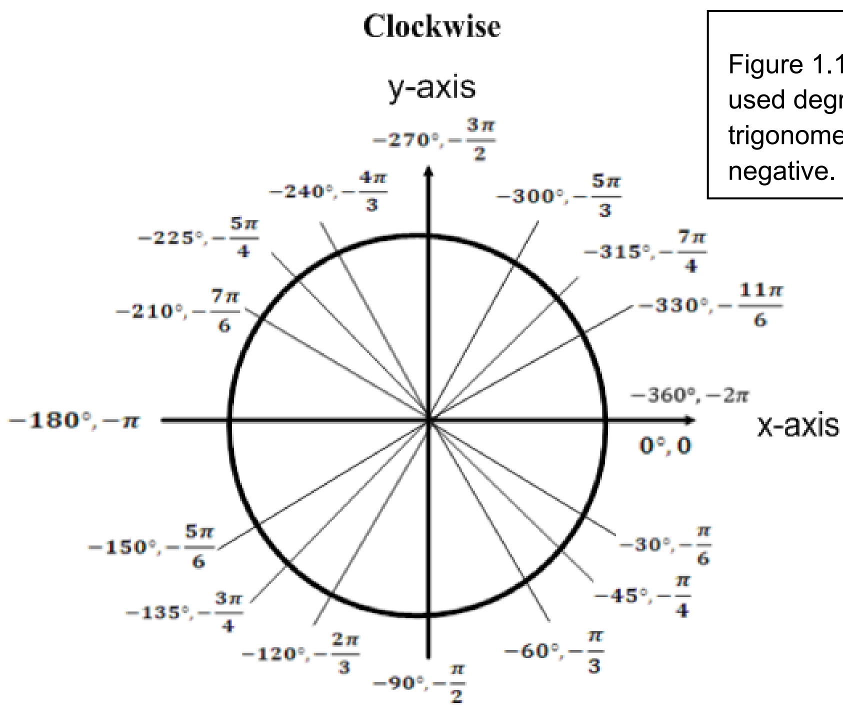


Figure 1.1.2d: Commonly used degree and radian in trigonometry where θ is negative.

Angles greater than $360°$ are created by the terminal side that makes more than one rotation ($360°=2\pi$). When the terminal side makes two complete rotations, it results in $720°$. So, as we think about angles in standard position, it makes sense that $360°$ and $720°$ are at the same standard position. ***Coterminal Angles*** are two angles with the same initial and terminal sides but differ by rotations and thus the two angles will result in the same standard position (See figure 1.1.2d).

EXAMPLES

Please work through the following examples before completing the 1.1.2 LEARNING ACTIVITY:

Example 1: Describe the standard position of $135°$.

$135°$ is in the second quadrant between $90°$ and $180°$ where the x-value is negative and the y-value is positive.

Example 2: Find the positive angle θ that is $0°<\theta<360°$ and is coterminal with $480°$.

Since $400°$ is more than $360°$ but less than $720°$, subtract 360 from 480

$480°-360°$

$=120°$

Thus, 120° and $400°$ are located at the same standard position.

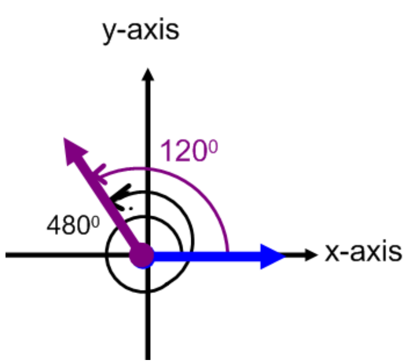


Figure 1.1.2d: Coterminal angles are angles in the same standard position but differ by rotation.

Example 3: Find the positive angle θ that is $0°<\theta<2\pi$ and is coterminal with $\frac{7\pi}{3}$.

Since $2\pi = 360°$ is one rotation, we can subtract 2π from $\frac{7\pi}{3}$.

$$\frac{7\pi}{3}-2\pi=\frac{7\pi}{3}-\frac{6\pi}{3}=\frac{\pi}{3}$$

Thus, $\frac{7\pi}{3}$ and $\frac{\pi}{3}$ are located at the same standard position.

1.1.2 LEARNING ACTIVITY

a. Describe the standard position of $315°$.

b. Find the positive angle θ that is $0°<\theta<2\pi$ and is coterminal with $\frac{11\pi}{3}$.

1.2 RIGHT TRIANGLE TRIGONOMETRY

Recall that the word Trigonometry means “measurement of triangles”. After learning how to convert between degrees and radians and angles in the standard position, we can now move to the study of right triangles.

1.2.1 PYTHAGOREAN THEOREM

Pythagorean Theorem (See table 1.2.1a) is used to find the sides of a right triangle. The side that is across from the $90°$ angle is always the hypotenuse which is the longest side of a right triangle. The other two sides are called the legs. We will denote the hypotenuse using the letter c, and the two legs can be denoted using letter *a* and *b* (See figure 1.2.1a). To find the missing side a right triangle, we will be given two sides to substitute into the formula $a^2+b^2=c^2$.

Table 1.2.1a

Pythagorean Theorem

$a^2+b^2=c^2$

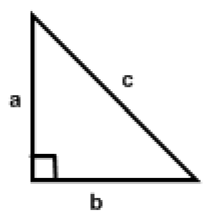
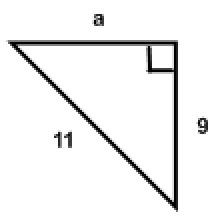


Figure 1.2.1a: A right triangle with hypotenuse denoted as letter c, and the two legs are denoted using letters a and b.

EXAMPLE

Please work through the following example before completing the 1.2.1 LEARNING ACTIVITY:

Find the length of the missing side of the right triangle.



c=11, and b=9 substitute into the formula

$9^2+b^2=11^2$

$81+b^2=121$ subtract 81 on both sides

$b^2=121-81$

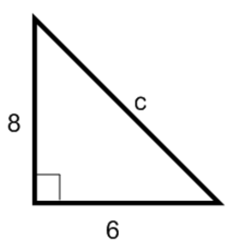
$b^2=40$ take a square root on both sides

$b=\sqrt{40}$ simplify the square root

$b=2\sqrt{10}$

1.2.1 LEARNING ACTIVITY

a. Find the length of the missing side of the right triangle.



1.2.2 TRIGONOMETRIC FUNCTIONS

Rather than just being called a function *f* or a function *g*, each of the six trigonometric functions actually have a name. The ***Six Trigonometric Functions*** are sine, cosine, tangent, cosecant, secant, and cotangent of an acute angle (See table 1.2.2a). The six trigonometric functions can be used to find angles or sides of a right triangle. The input of these functions is an acute angle θ of a right triangle, and the outputs are the ratios of the lengths of sides of right triangles. Figure 1.2.2a illustrates a right triangle with one acute angle labeled as θ. The side that is across from the right angle is the hypotenuse. The position of other two sides are named relatively to the acute angle θ.

Table 1.2.2a

Name	Abbreviation	Name	Abbreviation
sine	sin	cosecant	csc
cosine	cos	secant	sec
tangent	tan	cotangent	cot

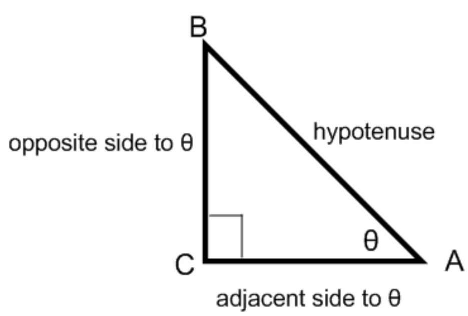


Figure 1.2.2a: Naming of a right triangle with a given acute angle θ.

The name for each of the six trigonometric functions represents the ratios of the lengths of sides of right triangles (See table 1.2.2b). For example, cosine of an acute angle θ is the length of the adjacent side to θ divided by the length of the hypotenuse. In the table 1.2.2b, notice that cosecant, secant, and cotangent are the reciprocal function to sine, cosine, and tangent respectively.

Table 1.2.2b

Definitions of Trigonometric Functions

$$\sin\theta = \frac{\text{opposite side to }\theta}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite side to }\theta}$$

$$\cos\theta = \frac{\text{adjacent side to }\theta}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side to }\theta}$$

$$\tan\theta = \frac{\text{opposite side to }\theta}{\text{adjacent side to }\theta} \qquad \cot\theta = \frac{\text{adjacent side to }\theta}{\text{opposite side to }\theta}$$

VIDEO 1.2.2A

Click here to watch similarity to define sine, cosine, and tangent.

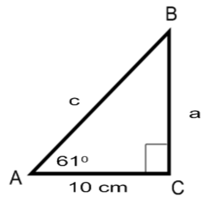
When solving problems using trigonometric functions, follow the steps below to determine which function to use:

Step 1: Label the hypotenuse which is across from the right angle.
Step 2: Locate the given acute angle.
Step 3: Label the adjacent side and opposite side in respect to the given acute angle.
Step 4: Choose an appropriate trigonometric function based on the given.

EXAMPLE

Please work through the following example before completing the 1.2.2 LEARNING ACTIVITY:

Find the sides *a* and *c* of a right triangle. Round your final answer to two decimal places.



By following the four steps that are listed above, side c is the hypotenuse, 10 cm is the adjacent side to 61°, and side *a* is the opposite side to 61°. Thus, tangent is the appropriate choice to find side *a*.

$$\tan\theta = \frac{\text{opposite side to }\theta}{\text{adjacent side to }\theta}$$

$$\tan 61° = \frac{a}{10}$$ use the cross product

$$a = \tan 61° \cdot 10$$ use a calculator in the degree mode

$$a = 18.04 \text{ cm}$$

To find side c, cosine is the appropriate function to use because we are given an adjacent side (10 cm) and need to find the hypotenuse (side *c*).

$$\cos\theta = \frac{\text{adjacent side to }\theta}{\text{hypotenuse}}$$

$$\cos 61° = \frac{10}{c}$$ use cross product

$$c \cdot \cos 61° = 10$$ divide both sides by cos 61°

$$c = \frac{10}{\cos 61°}$$ use a calculator in the degree mode

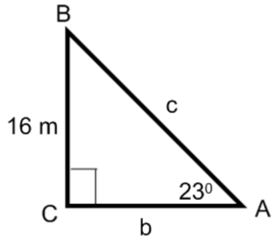
$$c = 20.63 \text{ cm}$$

VIDEO 1.2.2B

Click [Basic Trigonometry] to watch an example of Pythagorean Theorem and how to apply trigonometric functions.

1.2.2 LEARNING ACTIVITY

a. Find the side *b* and *c* of a right triangle. Round the answer to two decimal places.



1.2.3 SPECIAL RIGHT TRIANGLES

A $45°$-$45°$ right triangle and a $30°$-$60°$ right triangle are the two special right triangles that we can use to find the coordinates of the angles in standard position listed in figure 1.1.2c. In figure 1.2.3a, we assume each leg of a $45° - 45°$ right triangle has a length equal to 1. By using the Pythagorean Theorem, the hypotenuse is equal to $\sqrt{2}$.

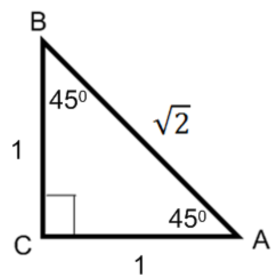


Figure 1.2.3a: A $45° - 45°$ right triangle with legs length equal to 1 and the hypotenuse equals to $\sqrt{2}$

A $30°$-$60°$ special right triangle also occurs frequently in the study of trigonometry. The $30°$-$60°$ special right triangle can be constructed from an equilateral triangle where each side has a length equal to 2 (See figure 1.2.3b). If we bisect the equilateral triangle, it will result a $30°$-$60°$ right triangle. Thus, a $30°$-$60°$ special right triangle has a hypotenuse of length 2 and a leg length of 1. Using the Pythagorean Theorem, we determine the side across from the $60°$ angle will equal the length of $\sqrt{3}$.

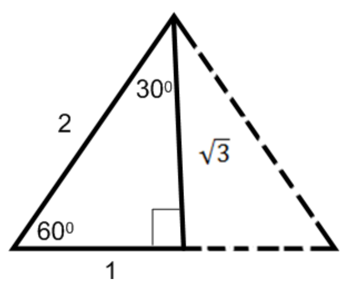


Figure 1.2.3b: A $30° - 60°$ right triangle with sides equal to 1 and $\sqrt{3}$. The hypotenuse equals to 2

EXAMPLES

Please work through the following two examples before completing the 1.2.3 LEARNING ACTIVITY:

Example 1: Find the six trigonometric function value of $45°$-$45°$ special right triangle.

$$\sin 45° = \frac{\text{opposite side to } 45°}{\text{hypotenuse}}$$

$$\sin 45° = \frac{1}{\sqrt{2}}$$ rationalize the denominator

VIDEO 1.2.3A

Click here to watch examples of how to rationalize the denominator. Video transcript available under Module 1 in Moodle.

$$\sin 45° = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\cos 45° = \frac{\text{adjacent side to } 45°}{\text{hypotenuse}}$$

$$\cos 45° = \frac{1}{\sqrt{2}} \qquad \text{rationalize the denominator}$$

$$\cos 45° = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\tan 45° = \frac{\text{opposite side to } 45°}{\text{adjacent side to } 45°}$$

$$\tan 45° = \frac{1}{1} = 1$$

$$\csc 45° = \frac{\text{hypotenuse}}{\text{opposite side to } 45°}$$

$$\csc 45° = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\sec 45° = \frac{\text{hypotenuse}}{\text{adjacent side to } 45°}$$

$$\sec 45° = \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\cot 45° = \frac{\text{adjacent side to } 45°}{\text{opposite side to } 45°}$$

$$\cot 45° = \frac{1}{1} = 1$$

Example 2: Find $\sin 60°$, $\cos 60°$ using $30°$-$60°$ special right triangle.

By looking at the figure 1.2.3b,

$$\sin 60° = \frac{\text{opposite side to } 60°}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{\text{adjacent side to } 60°}{\text{hypotenuse}} = \frac{1}{2}$$

VIDEO 1.2.3B

Click Basic Trigonometry II to watch examples of applying trigonometric functions and special right triangles.

1.2.3 LEARNING ACTIVITY

a. Find the exact value of $\tan 60°$, $\sin 30°$, $\cos 30°$, $\tan 30°$.

1.2.4 FUNDAMENTAL IDENTITIES

Many relationships exist among the six trigonometric functions that were listed in table 1.2.2b. These relationships are described as trigonometric identities. Trigonometric identities are the study of reciprocal identities, quotient identities, Pythagorean identities, and cofunction identities. In this section, we will begin to examine each identity.

Recall cosecant, secant, and cotangent is the reciprocal function in respect to sine, cosine, and tangent. We can establish the reciprocal identity for each of the six trigonometric functions by simply taking the reciprocal.

Let's take cosecant for example. We know that $\csc\theta = \frac{\csc\theta}{1} = \frac{\text{hypotenuse}}{\text{opposite side to }\theta}$, if we take a reciprocal on both sides of $\frac{\csc\theta}{1} = \frac{\text{hypotenuse}}{\text{opposite side to }\theta}$, we get $\frac{1}{\csc\theta} = \frac{\text{opposite side to }\theta}{\text{hytotenuse}} = \sin\theta$.Thus, all the reciprocal identities (See table 1.2.4a) can be established by following the above steps.

Table 1.2.4a

Reciprocal Identities

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

Table 1.2.4b

Quotient Identities

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Quotient identities (See table 1.2.4b) refer to trigonometric functions tangent and cotangent. Let's take tangent for example. If we take $\frac{\sin\theta}{\cos\theta}$, it can be written as $\frac{\frac{\text{opposite side to }\theta}{\text{hypotenuse}}}{\frac{\text{adjacent side to }\theta}{\text{hypotenuse}}}$. So, rewrite it as $\frac{\text{opposite side to }\theta}{\text{hypotenuse}} \times \frac{\text{hypotenuse}}{\text{adjacent side to }\theta}$, and we get $\frac{\text{opposite side to }\theta}{\text{adjacent side to }\theta}$, which is equal to $\tan\theta$. Therefore, $\tan\theta = \frac{\sin\theta}{\cos\theta}$.

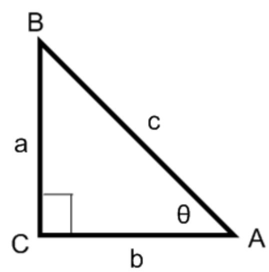
Another quotient identity which is $\cot\theta = \frac{\cos\theta}{\sin\theta}$ can be established by following the above approach.

Pythagorean Identities (See table 1.2.4c) can be derived from Pythagorean Theorem. By looking at the triangle below, we know that side *a* is the opposite side of θ, side *b* is the adjacent side of θ, and side *c* is the hypotenuse.

Table 1.2.4c

Pythagorean Identities

$$sin^2\theta + cos^2\theta = 1 \quad 1 + tan^2\theta = sec^2\theta \quad 1 + cot^2\theta = csc^2\theta$$



Starting with the Pythagorean Theorem, $a^2+b^2=c^2$, we can divide every term by c^2 and get $\frac{a^2}{c^2}+\frac{b^2}{c^2}=1$. We will then rewrite it as $\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1$. Since side *a* is the opposite side of θ and side *b* is the adjacent side of θ, and side *c* is the hypotenuse, we can write $\frac{a^2}{c^2}+\frac{b^2}{c^2}=1$ as $\left(\frac{\text{opposite}}{\text{hypotenuse}}\right)^2+\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)^2=1$. Since $\sin\theta=\frac{\text{opposite side to }\theta}{\text{hypotenuse}}$ and $\cos\theta=\frac{\text{adjacent side to }\theta}{\text{hypotenuse}}$, we can write $\left(\frac{\text{opposite}}{\text{hypotenuse}}\right)^2+\left(\frac{\text{adjacent}}{\text{hypotenuse}}\right)^2=1$ as $(\sin\theta)^2+(\cos\theta)^2=1$. So, $\sin^2\theta+\cos^2\theta=1$.

Two additional equivalent identities can be obtained by dividing every term by a^2 or b^2 respectively rather than c^2. The three identities together are the Pythagorean Identities.

EXAMPLES

Please work through the following two examples before completing the 1.2.4 LEARNING ACTIVITY:

Example 1: Let θ be an acute angle and $\sin\theta$ is given. Use the Pythagorean Identities to find $\cos\theta$

$$\sin\theta=\frac{\sqrt{21}}{5}$$

$$\sin^2\theta+\cos^2\theta=1$$

$\left(\frac{\sqrt{21}}{5}\right)^2 + \cos^2\theta = 1$ subtract $\left(\frac{\sqrt{21}}{5}\right)^2$ on both sides

$\cos^2\theta = 1 - \left(\frac{\sqrt{21}}{5}\right)^2$ simplify $\left(\frac{\sqrt{21}}{5}\right)^2$

$\cos^2\theta = 1 - \frac{21}{25}$

$\cos^2\theta = \frac{4}{25}$ take a square root on both sides

$\cos\theta = \frac{2}{5}$

Example 2: Use an identity to find the value of the expression.

$\cos 43° \cdot \sec 43°$

Use reciprocal identity.

$\cos 43° \cdot \sec 43°$ note $\sec\theta = \frac{1}{\cos\theta}$

$= \cos 43° \cdot \frac{1}{\cos 43°} = 1$

1.2.4 LEARNING ACTIVITY

a. Let θ be an acute angle and $\sin\theta$ is given. Use the Pythagorean Identities to find $\cos\theta$.

$\sin\theta = \frac{7}{8}$

b. Use an identity to find the value of the expression.

$\sin\frac{\pi}{2} \cdot \csc\frac{\pi}{2}$

1.2.5 COFUNCTION IDENTITIES

Cofunction identities demonstrate a unique relationship of two complementary angles of a right triangle. The value of a trigonometric function of θ is equal to the cofunction of the complement of θ. In other words, the angles of trigonometric functions are complements. (See table 1.2.5a).

Table 1.2.5a

Cofunction Identities

$\sin\theta = \cos(90° - \theta)$ $\cos\theta = \sin(90° - \theta)$

$\tan\theta = \cot(90° - \theta)$ $\cot\theta = \tan(90° - \theta)$

$\sec\theta = \csc(90° - \theta)$ $\csc\theta = \sec(90° - \theta)$

Note:

If θ is in radian, then replace $90°$ with $\frac{\pi}{2}$

VIDEO 1.2.5A

Click here to watch the relationship between cosine and sine of complements.

EXAMPLE

Please work through the following example before completing the 1.2.5 LEARNING ACTIVITY:

Find a cofunction with the same value as the given expression

$\sin 20°$

$\sin 20° = \cos(90°\text{-}20°)$

$= \cos 70°$

So, $\sin 20° = \cos 70°$

VIDEO 1.2.5B

Click here to watch sine and cosine complements examples.

1.2.5 LEARNING ACTIVITY

a. Find a cofunction with the same value as the given expression.

$\csc 40°$

1.2.6 FIND TRIGONOMETRY FUNCTION VALUE IN DECIMAL FORM

Given an angle of a right triangle, a graphing calculator can be used to find the trigonometry function value in decimal form. ***Trigonometry function value*** is the ratio of a right triangle in decimal form. Recall example 2 from 1.2.3 stated $\sin 60° = \frac{\text{opposite side to } 60°}{\text{hypotenuse}} = \frac{\sqrt{3}}{2}$. In another word, the exact answer of $\sin 60°$is $\frac{\sqrt{3}}{2}$ and the approximated answer when dividing $\sqrt{3}$ by 2 (in decimal form) is 0.8660. When we enter degree for an angle in the calculator, set the calculator in degree mode (See figure 1.2.6a). When we enter radian for an angle, set the calculator in radian mode (See figure 1.2.6b).

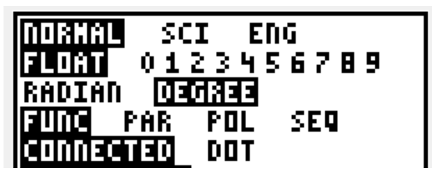


Figure 1.2.6a: When enter degree for an angle, set the calculator in degree mode.

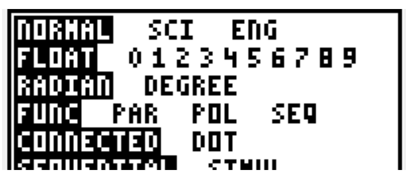


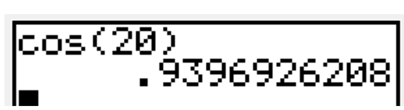
Figure 1.2.6b: When enter radian for an angle, set the calculator in radian mode.

EXAMPLES

Please work through the following four examples before completing the 1.2.6 LEARNING ACTIVITY:

Example 1: Use a calculator to find the value of trigonometric function $\cos 20°$ and round the final answer to four decimal places.

$\cos 20° = 0.9397$ in degree mode



Example 2: Use a calculator to find the value of trigonometric function $\sin 80°$ and round the final answer to four decimal places.

$\sin 80° = 0.9848$ in degree mode

Example 3: Use a calculator to find the value of trigonometric function $\sec 56°$ and round the final answer to four decimal places.

$\sec 56° = \dfrac{1}{\cos 56°} = 1.7883$ in degree mode

Example 4: Use a calculator to find the value of trigonometric function $\sin \frac{3\pi}{10}$ and round the final answer to four decimal places.

$\sin \dfrac{3\pi}{10} = 0.8090$ in radian mode

VIDEO 1.2.6A

Click [Example: Calculator to evaluate a trig function] for a demonstration of using a graphing calculator to evaluate a trigonometric function.

1.2.6 LEARNING ACTIVITY

a. Use a calculator to find the value of trigonometric function and round the final answer to four decimal places.

$\csc 40°$

b. Use a calculator to find the value of trigonometric function and round the final answer to four decimal places.

$\sin 120°$

c. Use a calculator to find the value of trigonometric function and round the final answer to four decimal places.

$\sin\frac{2\pi}{3}$

1.2.7 FIND THE ACUTE ANGLE

Given a trigonometric function value, we will use an inverse trigonometric key on the graphing calculator such as $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ to find the acute angle. The acute angle can either be measured in degree or in radian. When the angle is measured in degree, we set the calculator in degree mode. When the angle is measured in radian, be sure to set the calculator in radian mode. If the acute angle is in radian, the graphing calculator will only display the decimal approximation answer rather than be expressed as a fraction using multiples of π.

EXAMPLE

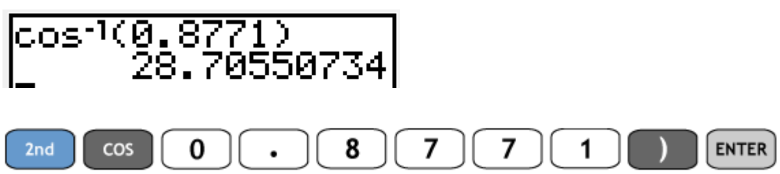
Please work through the following two examples before completing the 1.2.7 LEARNING ACTIVITY:

Example 1: Use a calculator to the value of the acute angle θ to the nearest degree.

$\cos\theta = 0.8771$

$\theta = \cos^{-1}(0.8771)$ in degree mode

$\theta = 29°$

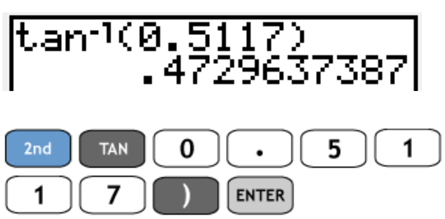


Example 2: Use a calculator to the value of the acute angle θ in radians, round to three decimal places.

$\tan\theta = 0.5117$

$\theta = \tan^{-1}(0.5117)$ in radian mode

$\theta = 0.473$



1.2.7 LEARNING ACTIVITY

a. Use a calculator to the value of the acute angle θ to the nearest degree.

$\tan\theta = 26.0307$

b. Use a calculator to the value of the acute angle θ in radians, round to three decimal places.

$\sin\theta = 0.9499$

1.3 TRIGONOMETRIC FUNCTION OF ANY ANGLE

The study of the trigonometric function of any angle is finding the exact value of the trigonometric functions. Given an acute angle θ in the standard position and a point on the x,y-axis, we will be able to find the exact value of all six trigonometric functions. In figure 1.3a, the angle θ is in standard position where θ lies in the first quadrant. The point $P=(x,y)$ is a point *r* units from the origin on the terminal side of θ. A right triangle is formed by drawing a line segment from *P* perpendicular to the x-axis. Thus, *y* is the opposite side to θ and *x* is the adjacent side to θ.

If the angle θ is not acute, then the angle will lie in the second, third, or fourth quadrant. When the angle θ is between $90°$ to $180°$, θ lies in the second quadrant (See figure 1.3b). When the angle θ is between $180°$ to $270°$, θ lies in the third quadrant (See figure 1.3c). When the angle θ is between $270°$ to $360°$, θ lies in the fourth quadrant (See figure 1.3d).

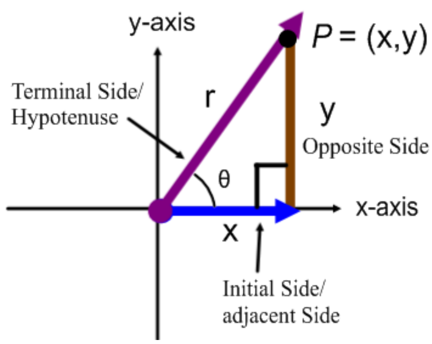


Figure 1.3a: θ lies in the first quadrant. $P=(x,y)$ is *r* units from the origin *y* is the opposite side to θ and *x* is the adjacent side to θ.

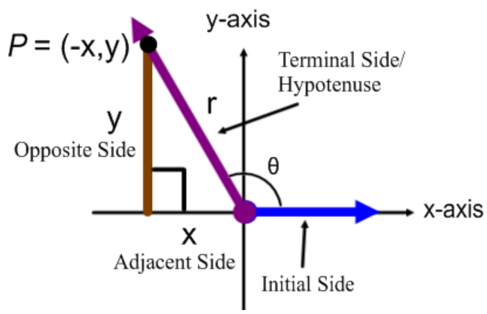


Figure 1.3b: θ lies in the second quadrant. $P=(-x,y)$ is *r* units from the origin *y* is the opposite side to θ and -*x* is the adjacent side to θ.

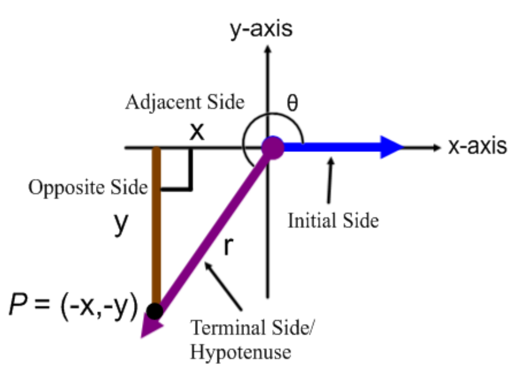


Figure 1.3c: θ lies in the third quadrant. $P=(-x,-y)$ is *r* units from the origin -*y* is the opposite side to θ and -*x* is the adjacent side to θ.

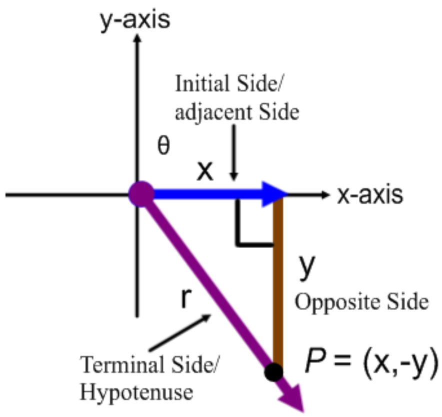


Figure 1.3d: θ lies in the fourth quadrant. P=(x,-y) is *r* units from the origin *-y* is the opposite side to θ and *x* is the adjacent side to θ.

1.3.1 TRIGONOMETRIC FUNCTIONS OF ANY ANGLE

Let θ be any angle in standard position and let P=(x,y) be a point on the terminal side of θ. If $r=\sqrt{x^2+y^2}$ is the distance from (0,0) to (x,y), the six trigonometric functions of any angle θ are defined in table 1.3.1a.

Table 1.3.1a

Definition of Trigonometric Function of Any Angle

$$\sin\theta = \frac{\text{opposite side to }\theta}{\text{hypotenuse}} = \frac{y}{r} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite side to }\theta} = \frac{r}{y}$$

$$\cos\theta = \frac{\text{adjacent side to }\theta}{\text{hypotenuse}} = \frac{x}{r} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent side to }\theta} = \frac{r}{x}$$

$$\tan\theta = \frac{\text{opposite side to }\theta}{\text{adjacent side to }\theta} = \frac{y}{x} \qquad \cot\theta = \frac{\text{adjacent side to }\theta}{\text{opposite side to }\theta} = \frac{x}{y}$$

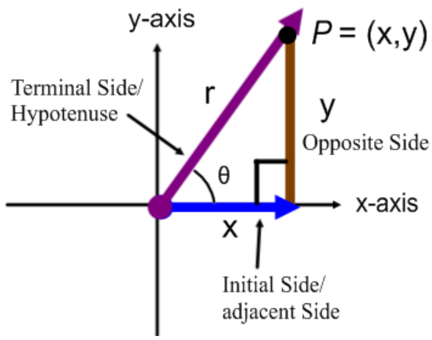
EXAMPLE

<u>Please work through the following two examples before completing the 1.3.1 LEARNING ACTIVITY:</u>

Example 1: Given a coordinate in the standard position. Find the exact value of each of the six trigonometric function of θ

$(3{,}7)$

Since $(3{,}7)$ **is in the first quadrant where** $x=3$ and $y=7$, **use figure 1.3a to find *r* by using the Pythagorean Theorem.**



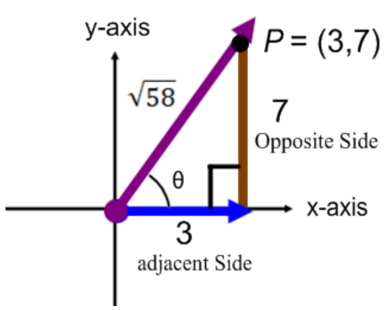
$r^2=x^2+y^2$

$r^2=(3)^2+(7)^2$ simplify

$r^2=58$ take a square root on both sides

$r=\sqrt{58}$

Thus,



Once the *r* is found, write out the six trigonometric functions

$\sin\theta=\frac{y}{r}=\frac{7}{\sqrt{58}}$ rationalize the denominator

$\sin\theta = \frac{7\sqrt{58}}{58}$

$\cos\theta = \frac{x}{r} = \frac{3}{\sqrt{58}}$ rationalize the denominator

$\cos\theta = \frac{3\sqrt{58}}{58}$

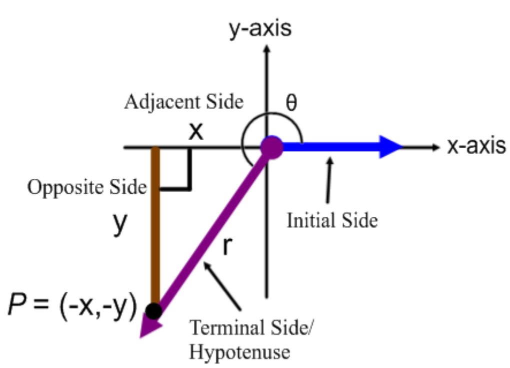
$\tan\theta = \frac{y}{x} = \frac{7}{3}$ $\csc\theta = \frac{r}{y} = \frac{\sqrt{58}}{7}$

$\sec\theta = \frac{r}{x} = \frac{\sqrt{58}}{3}$ $\cot\theta = \frac{x}{y} = \frac{3}{7}$

Example 2: Given a trigonometric ratio, find the exact value of each of the remaining trigonometric functions of θ.

$\sin\theta = -\frac{3}{5}$, θ in third quadrant

Since θ is in the third quadrant, we will use figure 1.3c. Notice that when θ is in the third quadrant, both x and y-values will be negative. So, $\sin\theta = -\frac{3}{5}$ **and** $\sin\theta = \frac{y}{r}$**, thus** $y=-3$ and $r=5$. **We will first apply the Pythagorean Theorem to find x. Keep in mind that θ is in the third quadrant, so the x-value will be negative as well.**



$r^2=x^2+y^2$

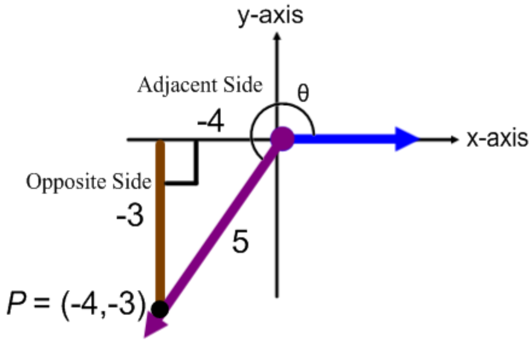
$5^2=x^2+(-3)^2$ simplify

$25=x^2+9$ subtract 9 on both sides

$16=x^2$ take a square root on both sides

$-4=x$ since θ is in the third quadrant

Thus,



Once we found the side x, write out the remaining trigonometric functions.

$$\cos\theta=\frac{x}{r}=\frac{-4}{5}=-\frac{4}{5},\quad \sec\theta=\frac{r}{x}=\frac{5}{-4}=-\frac{5}{4},\quad \tan\theta=\frac{y}{x}=\frac{-3}{-4}=\frac{3}{4},$$

$$\cot\theta=\frac{x}{y}=\frac{-4}{-3}=\frac{4}{3},\quad \csc\theta=\frac{r}{y}=\frac{5}{-3}=-\frac{5}{3}$$

1.3.1 LEARNING ACTIVITY

a. Given a coordinate in the standard position. Find the exact value of each of the six trigonometric functions of θ.

(5,-5)

b. Given a trigonometric ratio, find the exact value of each of the remaining trigonometric functions of θ.

$\cos\theta=\frac{4}{5}$, θ is in the fourth quadrant

1.3.2 REFERENCE ANGLE

Let θ be a nonacute angle in standard position that lies in a quadrant. A ***Reference Angle*** is a positive acute angle θ' formed by the terminal side and the x-axis. If θ lies in the second quadrant, the reference angle called θ'(theta prime) can be found by $180° - \theta$ (See figure 1.3.2a). If θ lies in the third quadrant, the reference angle can be found by $\theta - 180°$ (See figure 1.3.2b). If θ lies in the fourth quadrant, the reference angle can be found by $360° - \theta$ (See figure 1.3.2c).

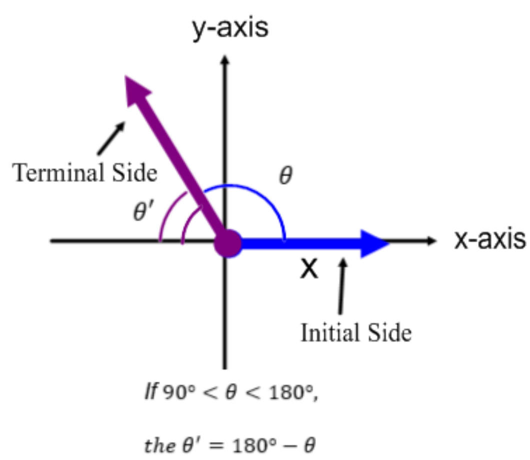


Figure 1.3.2a: θ lies in the second quadrant, its reference angle can be found by $180° - \theta$.

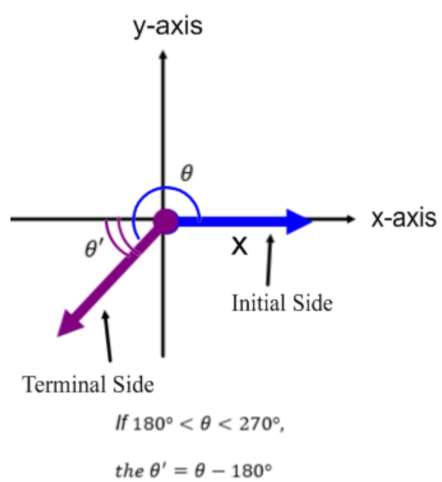


Figure 1.3.2b: θ lies in the third quadrant, its reference angle can be found by $\theta - 180°$.

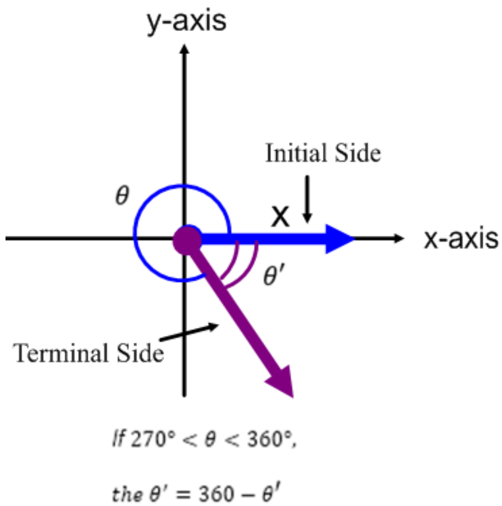


Figure 1.3.2c: θ lies in the fourth quadrant, its reference angle can be found by $360° - \theta$.

EXAMPLES

Please work through the following two examples before completing the 1.3.2 LEARNING ACTIVITY:

Example 1: Find the reference angle.

$150°$

$\theta' = 180° - 150° = 30°$

Example 2: Find the reference angle.

$\frac{4\pi}{3}$

$\theta' = \frac{4\pi}{3} \cdot \frac{180°}{\pi} = 240°$ convert to degree if it helps to locate where $\frac{4\pi}{3}$ is

$\theta' = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$ since $240°$ is in the third quadrant and $\pi = 180°$, subtract π

1.3.2 LEARNING ACTIVITY

a. Find the reference angle.

$225°$

b. Find the reference angle.

$\frac{3\pi}{4}$

1.4 TRIGONOMETRIC FUNCTION OF REAL NUMBERS

Trigonometric function of real numbers is about finding the coordinates on the unit circle with a given angle. Review figure 1.1.2c below that contains all the commonly used degree and radian in trigonometry. In this section, we are going to compute the coordinates for any of these angles.

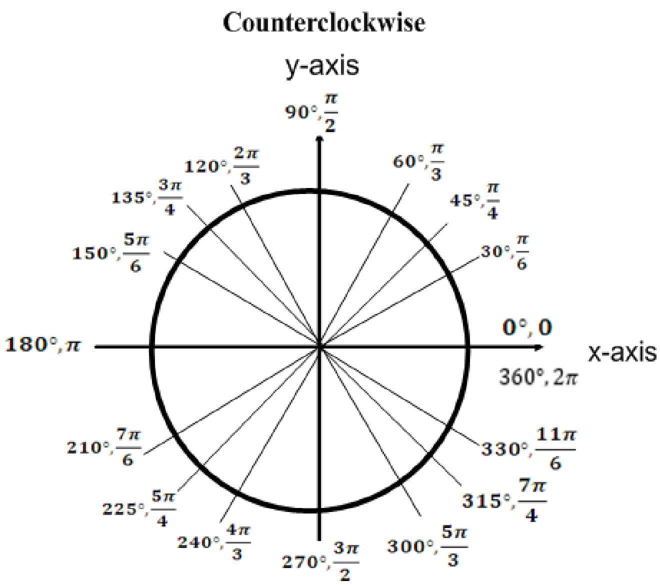


Figure 1.1.2c: Commonly used degree and radian in trigonometry where θ is positive.

1.4.1 UNIT CIRCLE

A ***Unit Circle*** is a circle of radius 1, where its center is located at the origin in a standard position. The equation of a unit circle is $x^2+y^2=1$. The central angle, which is created by two radiuses and the intercept arc, is measured in *t* radians. In addition, in a unit circle, the radian measure of central angle is equal to the length of the intercepted arc.

Let $P=(x,y)$ denoted as a point on the unit circle that has arc length *t* from the coordinate $(1,0)$. If the *t* radian is positive, point *P* is reached by moving counterclockwise from $(1,0)$(See figure 1.4.1a). If the *t* radian is negative, point *P* is reached by moving clockwise from $(1,0)$(See figure 1.4.1b).

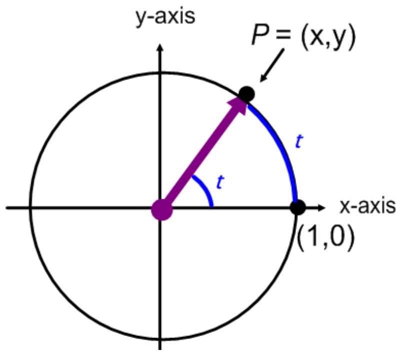


Figure 1.4.1a: *t* radian is positive, point *P* is reached by moving counterclockwise from $(1{,}0)$.

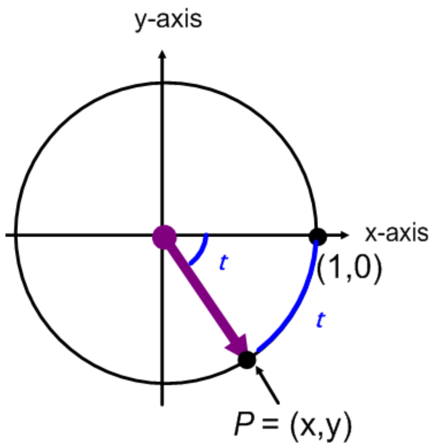


Figure 1.4.1b: the *t* radian is negative, point *P* is reached by moving clockwise from $(1{,}0)$.

If t is a real number and P=(x,y) is a point on the unit circle that corresponds to t, then trigonometric functions in terms of a unit circle are defined in table 1.4.1a.

Table 1.4.1a

Definitions of the Trigonometric Functions in Terms of a Unit Circle

$\sin t = y$	$\csc t = \frac{1}{y}, y \neq 0$
$\cos t = x$	$\sec t = \frac{1}{x}, x \neq 0$
$\tan t = \frac{y}{x}$	$\cot t = \frac{x}{y}, y \neq 0$

Recall table 1.3.1a where $\sin\theta = \frac{\text{opposite side to }\theta}{\text{hypotenuse}} = \frac{y}{r}$. Since the radius on the unit circle is 1, we can say $r = 1$. Thus, $\sin t = \frac{y}{r} = \frac{y}{1} = y$. In other words, the definitions of the trigonometric functions in terms of a unit circle is simply saying sine of any angle *t* is the y-value of the coordinate P=(x,y). Cosine of any angle *t* is the x-value of the coordinate P=(x,y). Tangent of any angle *t* is the y-value divided by the x-value of the coordinate P=(x,y).

EXAMPLES

Please work through the following three examples before completing the 1.4.1 LEARNING ACTIVITY:

Example 1: Find the exact value of the trigonometric functions at t=$\frac{\pi}{6}$.

$\frac{\pi}{6}\cdot\frac{180°}{\pi}=30°$ convert to degree if it helps to locate where $\frac{\pi}{6}$ is

So, t=30°

Recall figure 1.2.3b, the 30°-60° special right triangle.

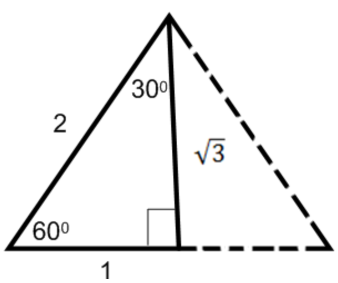


Figure 1.2.3b: A $30° - 60°$ right triangle with sides equal to 1 and $\sqrt{3}$. The hypotenuse equals to 2.

Since the radius on the unit circle has radius 1, divide every side by 2. The hypotenuse will be 1, the side opposite to 60° is $\frac{\sqrt{3}}{2}$, and the side opposite to 30° is $\frac{1}{2}$ (See figure 1.4.1c).

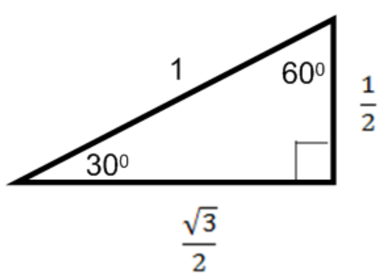


Figure 1.4.1c. The hypotenuse is 1, the side opposite to $60°$ is $\frac{\sqrt{3}}{2}$, and the side opposite to $30°$ is $\frac{1}{2}$.

Now draw the special right triangle on the unit circle (See figure 1.4.1d).

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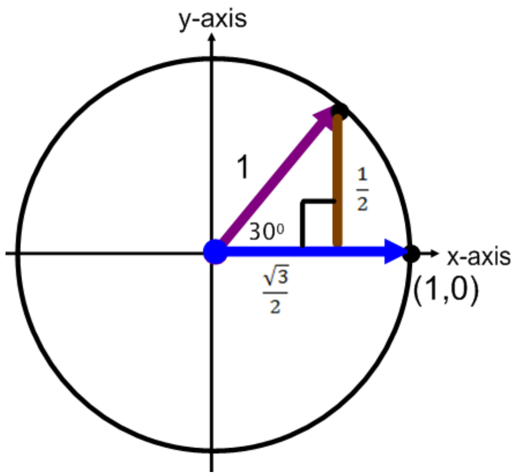
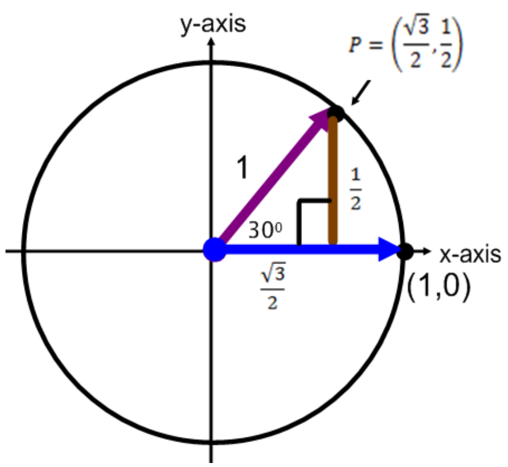


Since t is a real number and P=(x,y) is a point on the unit circle that corresponds to t, then P=$\left(\frac{\sqrt{3}}{2},\frac{1}{2}\right)$, which means point P is $\frac{\sqrt{3}}{2}$ unit to the right and $\frac{1}{2}$ unit above the origin.



So, trigonometric functions in terms of a unit circle are

$$\sin t = y, \quad \sin\frac{\pi}{6} = \frac{1}{2}$$

$$\cos t = x, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\tan t = \frac{y}{x}, \quad \tan\frac{\pi}{6} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$ divide numerator by denominator

$\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$ rationalized the denominator

$\csc t=\frac{1}{y}, y\neq 0,\quad \csc t=\frac{1}{\frac{1}{2}}=2$

$\sec t=\frac{1}{x}, x\neq 0,\quad \sec t=\frac{1}{\frac{\sqrt{3}}{2}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$ rationalized the denominator

$\cot t=\frac{x}{y}, y\neq 0,\quad \cot t=\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}=\sqrt{3}$

Example 2: Find the exact value of the trigonometric functions at $t=\frac{\pi}{4}$.

$\frac{\pi}{4}\cdot\frac{180°}{\pi}=45°$ convert to degree if it helps to locate where $\frac{\pi}{4}$ is

So, $t=45°$

Recall figure 1.2.3a, the $45°$-$45°$ special right triangle.

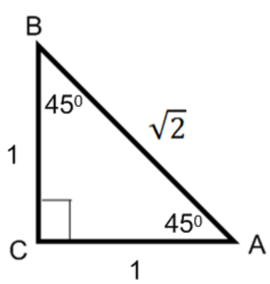


Figure 1.2.3a: A$45° - 45°$ right triangle with legs length equal to 1 and the hypotenuse equals to $\sqrt{2}$.

Since the radius on the unit circle has radius 1, divide every side by $\sqrt{2}$, then the hypotenuse will be 1 and the other two sides will be $\frac{\sqrt{2}}{2}$.

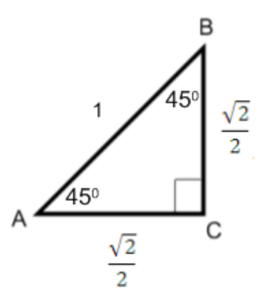
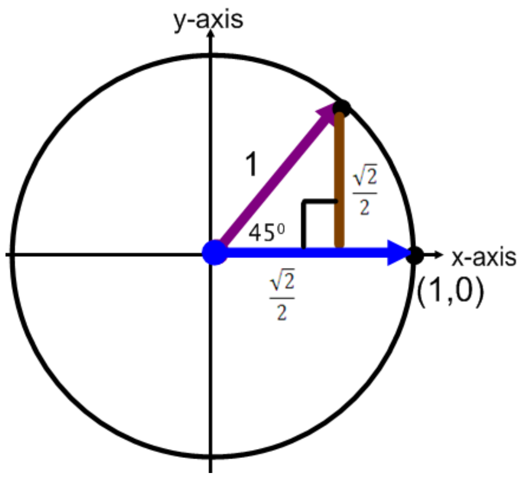
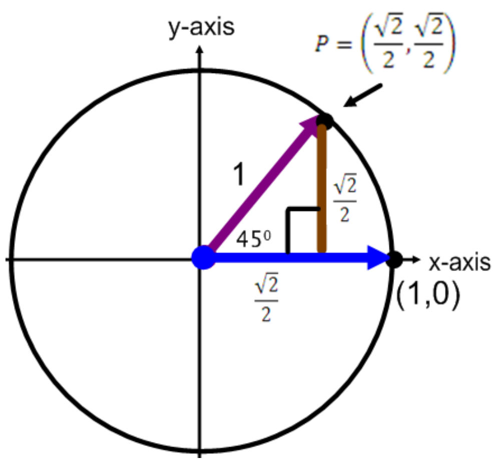


Figure 1.4.1d**.** The hypotenuse is 1, the side opposite to $45°$ is $\frac{\sqrt{2}}{2}$, and the side opposite to $45°$ is $\frac{\sqrt{2}}{2}$.

Now draw the $45°$-$45°$ special right triangle on the unit circle.



Since t is a real number and $P=(x,y)$ is a point on the unit circle that corresponds to t, then $P=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$.



So, trigonometric functions in terms of a unit circle are

$$\sin t=y,\quad \sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

$$\cos t=x,\quad \cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$$

$$\tan t=\frac{y}{x},\quad \tan\frac{\pi}{4}=\frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=1$$

$$\csc t = \frac{1}{y}, y \neq 0, \quad \csc t = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\sec t = \frac{1}{x}, x \neq 0, \quad \sec t = \frac{1}{\frac{\sqrt{2}}{2}} = \frac{2}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$$\cot t = \frac{x}{y}, y \neq 0, \quad \cot t = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1$$

Example 3: Find the exact value of the trigonometric functions $\sin\frac{2\pi}{3}$ and $\cos\frac{2\pi}{3}$.

$\frac{2\pi}{3} \cdot \frac{180°}{\pi} = 120°$ convert to degree if it helps to locate where $\frac{2\pi}{3}$ is

So, $t = 120°$

Note that $120°$ is located in the quadrant II, thus the x-value is negative and the y-value is positive.

Now, we will begin by finding the reference angle of $120°$

$\theta' = 180° - 120° = 60°$

Since the reference angle of $\frac{2\pi}{3}$ or $120°$ is $60°$, use the $30°$-$60°$ special right triangle to find the $P = (-x, y)$ (See figure 1.4.1e)

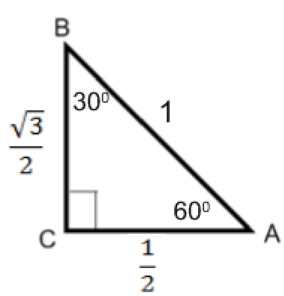
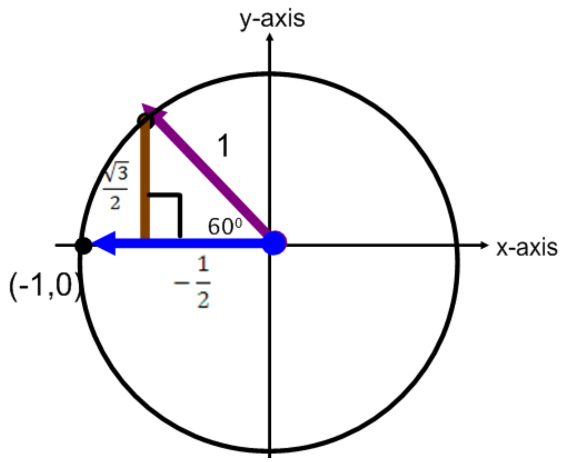


Figure 1.4.1e**.** The hypotenuse is 1, the side opposite to $60°$ is $\frac{\sqrt{3}}{2}$, and the side opposite to $30°$ is $\frac{1}{2}$.

Now draw the special right triangle on the unit circle, since t is a real number and $P = (x, y)$ is a point on the unit circle that corresponds to t, then $P = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.



So, trigonometric functions in terms of a unit circle are

$$\sin t = y, \quad \sin\frac{2\pi}{3} \text{ or } 120° = \frac{\sqrt{3}}{2}$$

$$\cos t = x, \quad \cos\frac{2\pi}{3} \text{ or } 120° = -\frac{1}{2}$$

VIDEO 1.4.1

Click the following videos for more examples of unit circles.

Unit circle definition of trig functions

Example: Unit circle definition of sin and cos

Example: Using the unit circle definition of trig functions

Example: Trig function values using unit circle definition

1.4.1 LEARNING ACTIVITY

a. Find the exact value of the trigonometric functions at $t = \frac{\pi}{3}$.

b. Find the exact value of the trigonometric functions $\sin\frac{5\pi}{4}$ and $\cos\frac{5\pi}{4}$.

MAJOR CONCEPTS

KEY CONCEPTS

Radians is another way to measure angles. Angles can be converted between degree and radian.

Pythagorean Theorem is used to find a missing side when two sides of a right triangle are given.

The six trigonometric functions can be used to find the missing sides of a right triangle when one side and one angle is given or find the missing angles of a right triangle when two sides are given.

The $45°$-$45°$, $30°$-$60°$ special right triangles and trigonometric functions of any angles are two key components to derive all the coordinates on the unit circle.

KEY TERMS

Angle	1 Radian	Coterminal Angle
Six Trigonometric Functions	Trigonometric function value	Reference Angle
Unit Circle		

GLOSSARY

An ***Angle*** is formed by two rays that have a common endpoint.

1 Radian is the measure of a central angle of a circle that intercept an arc where the arc length is the same as radius.

Coterminal Angles are two angles with the same initial and terminal sides but differ by rotations and thus the two angles will result in the same standard position.

The ***Six Trigonometric Functions*** are sine, cosine, tangent, cosecant, secant, and cotangent of an acute angle.

Trigonometry function value is the ratio of a right triangle in decimal form.

A ***Reference Angle*** is a positive acute angle θ' formed by the terminal side and the x-axis.

A ***Unit Circle*** is a circle of radius 1, where its center is located at the origin in a standard position.

ASSESSMENT

ANSWERS TO LEARNING ACTIVITIES

1.1.1 LEARNING ACTIVITIES

a. Convert 330° to radians. Express the final answer as a multiple of π.

Answer: $\frac{11\pi}{6}$ **radian**

b. Convert $\frac{3\pi}{4}$ radians to degrees.

Answer: $135°$

1.1.2 LEARNING ACTIVITIES

a. Describe the standard position of $315°$.

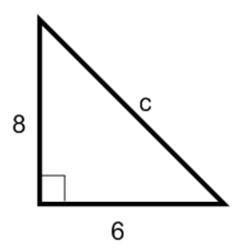
Answer: $315°$ is in the fourth quadrant between $270°$ and $360°$ where the x-value is positive and the y-value is negative.

b. Find the positive angle θ that are $0° < \theta < 2\pi$ that is coterminal with $\frac{11\pi}{3}$.

Answer: $\frac{5\pi}{3}$

1.2.1 LEARNING ACTIVITY

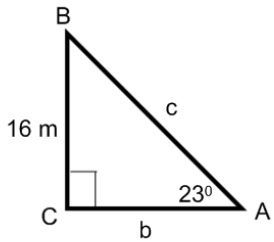
a. Find the length of the missing side of the right triangle.



Answer: $c=10$

1.2.2 LEARNING ACTIVITY

a. Find the side *b* and *c* of a right triangle. Round the answer to two decimal places.



Answer: b=37.69, c=40.95

1.2.3 LEARNING ACTIVITY

a. Find the exact value of $\tan 60°$, $\sin 30°$, $\cos 30°$, $\tan 30°$.

Answer: $\tan 60°=\sqrt{3}$, $\sin 30°=\frac{1}{2}$, $\cos 30°=\frac{\sqrt{3}}{2}$, $\tan 30°=\frac{\sqrt{3}}{3}$

1.2.4 LEARNING ACTIVITY

a. Let θ be an acute angle and $\sin\theta$ is given. Use the Pythagorean Identities to find $\cos\theta$.

$$\sin\theta=\frac{7}{8}$$

Answer: $\cos\theta=\frac{\sqrt{15}}{8}$

b. Use an identity to find the value of the expression.

$$\sin\frac{\pi}{2}\cdot\csc\frac{\pi}{2}$$

Answer: 1

1.2.5 LEARNING ACTIVITY

a. Find a cofunction with the same value as the given expression.

$$\csc 40°$$

Answer: $\sec 50°$

1.2.6 LEARNING ACTIVITY

a. Use a calculator to find the value of trigonometric function and round the final answer to four decimal places.

$\csc 40°$

Answer: 1.5557

b. Use a calculator to find the value of trigonometric function and round the final answer to four decimal places.

$\sin 120°$

Answer: 0.8660

c. Use a calculator to find the value of trigonometric function and round the final answer to four decimal places.

$\sin \frac{2\pi}{3}$

Answer: 0.8660

1.2.7 LEARNING ACTIVITY

a. Use a calculator to the value of the acute angle θ to the nearest degree.

$\tan\theta = 26.0307$

Answer: $\theta = 88°$

b. Use a calculator to the value of the acute angle θ in radians, round to three decimal places.

$\sin\theta = 0.9499$

Answer: $\theta = 1.253$ rad

1.3.1 LEARNING ACTIVITY

a. Given a coordinate in the standard position. Find the exact value of each of the six trigonometric function of θ.

(5,-5)

Answer: $\sin\theta = -\frac{\sqrt{2}}{2}, \cos\theta = \frac{\sqrt{2}}{2}, \tan\theta = -1, \csc\theta = -\sqrt{2}, \sec\theta = \sqrt{2}, \cot\theta = -1$

b. Given a trigonometric ratio, find the exact value of each of the remaining trigonometric function of θ.

$\cos\theta = \frac{4}{5}$, θ is in the fourth quadrant

Answer: $\sin\theta = -\frac{3}{5}, \tan\theta = -\frac{3}{4}, \csc\theta = -\frac{5}{3}, \sec\theta = \frac{5}{4}, \cot\theta = -\frac{4}{3}$

1.3.2 LEARNING ACTIVITY

a. Find the reference angle.

225°

Answer: $\theta' = 45°$

b. Find the reference angle.

$\frac{3\pi}{4}$

Answer: $\theta' = \frac{\pi}{4}$

1.4.1 LEARNING ACTIVITY

a. Find the exact value of the trigonometric functions at $t = \frac{\pi}{3}$.

Answer: $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \cos\frac{\pi}{3} = \frac{1}{2}, \tan\frac{\pi}{3} = \sqrt{3}, \csc\frac{\pi}{3} = \frac{2\sqrt{3}}{3}, \sec\frac{\pi}{3} = 2, \cot\frac{\pi}{3} = \frac{\sqrt{3}}{3}$

b. Find the exact value of the trigonometric functions $\sin\frac{5\pi}{4}$ and $\cos\frac{5\pi}{4}$.

Answer: $\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}, \cos\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$

MODULE REINFORCEMENT

SHORT ANSWER QUESTIONS

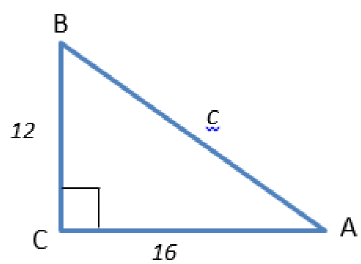
1) Convert the angle in degree to radians. Express your answer as a multiple of π. State the answer in fraction form.

$$30°$$

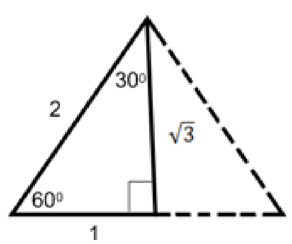
2) Find a positive angle less than $360°$ that is coterminal with the given angle.

$$640°$$

3) Use the Pythagorean Theorem to find the missing side *c*.



4) Use the $30°$-$60°$ special right triangle to find exact value of $\sec 30°$. Rationalize the denominator if needed.



5) Let θ be an acute angle and $\sin\theta$ is given. Use the Pythagorean Identities to find $\cos\theta$.

$$\sin\theta = \frac{\sqrt{3}}{5}$$

6) Use a calculator to find the value of trigonometric function and round the final answer to four decimal places.

$$\cot\frac{\pi}{9}$$

7) Use a calculator to find the value of the acute angle θ to the nearest degree.

$$\tan\theta = 4.6252$$

8) Given a coordinate in the standard position. Find the exact value of $\sin\theta$ and $\cos\theta$. Rationalize the denominator if needed.

$$(3,-3)$$

9) Given a trigonometric ratio, find the exact value of $\tan\theta$.

$$\cos\theta = -\frac{3}{5}, \theta \text{ is in the third quadrant}$$

10) Find the exact value of the trigonometric function.

$$\cos\frac{7\pi}{4} \text{ and } \sin\frac{7\pi}{4}$$

MULTIPLE CHOICE: READ THE FOLLOWING QUESTIONS OR STATEMENTS AND SELECT THE BEST ANSWER.

1) __________ Convert the angle in radians to degrees.

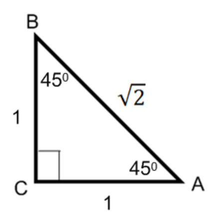
$$\frac{10\pi}{9} \text{ radian}$$

a. $1.1111°$
b. $0.00617°$
c. $200°$
d. $422.222°$

2) __________ Find the positive angle θ, $0° < \theta < 2\pi$ that is coterminal with $\frac{8\pi}{3}$.

a. $\frac{2\pi}{3}$
b. $\frac{5\pi}{3}$
c. 2π
d. $\frac{\pi}{3}$

3) __________ Use the $45°$-$45°$ special right triangle to find exact value of $\csc 45°$. Rationalize the denominator if needed.



a. 1
b. $\sqrt{2}$
c. $\frac{\sqrt{2}}{2}$
d. 2

4) __________ Use an identity to find the value of the expression.

$$\cos 44° \cdot \sec 44°$$

a. $44°$
b. $136°$
c. $46°$
d. 1

5) __________ Find a cofunction with the same value as the given expression.

$$\cot 50°$$

a. $\tan 50°$
b. $\tan 40°$
c. $\tan 130°$
d. $\tan 230°$

6) __________ Use a calculator to find the value of trigonometric function and round the final answer to four decimal places.

$$\sin 10°$$

a. -0.5440
b. $80°$
c. 0.1736
d. 0.9848

VIDEO

Click [Basic Trigonometry] for review.

7) __________ Use a calculator to find the value of trigonometric function and round the final answer to four decimal places.

$$\cos \frac{\pi}{10}$$

a. 0.9511
b. 0.9999
c. $56.5487°$
d. $103.4513°$

8) __________ Use a calculator to find the value of the acute angle θ in degrees and round the final answer to three decimal places.

$$\cos\theta = 0.4112$$

a. $0.9999°$
b. $65.720°$
c. $1.1470°$

9) __________ Find the reference angle.

$$300°$$

a. $30°$
b. $300°$
c. $120°$
d. $60°$

10) __________ Find the exact value of the trigonometric function.

$$\cos\frac{5\pi}{3}$$

a. $\frac{\sqrt{3}}{2}$
b. $\frac{\sqrt{2}}{2}$
c. $\frac{1}{2}$

ANSWER KEY

Short Answer Questions	Multiple Choice
1) $\frac{\pi}{6}$ 2) 280° 3) c=20 4) $\frac{2\sqrt{3}}{3}$ 5) $\cos\theta = \frac{\sqrt{22}}{5}$ 6) 2.7475 7) θ=78° 8) $\sin\theta = -\frac{\sqrt{2}}{2}, \cos\theta = \frac{\sqrt{2}}{2}$ 9) $\tan\theta = \frac{4}{3}$ 10) $\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2},\ \sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$	1) c 2) a 3) b 4) d 5) b 6) c 7) a 8) b 9) d 10) c

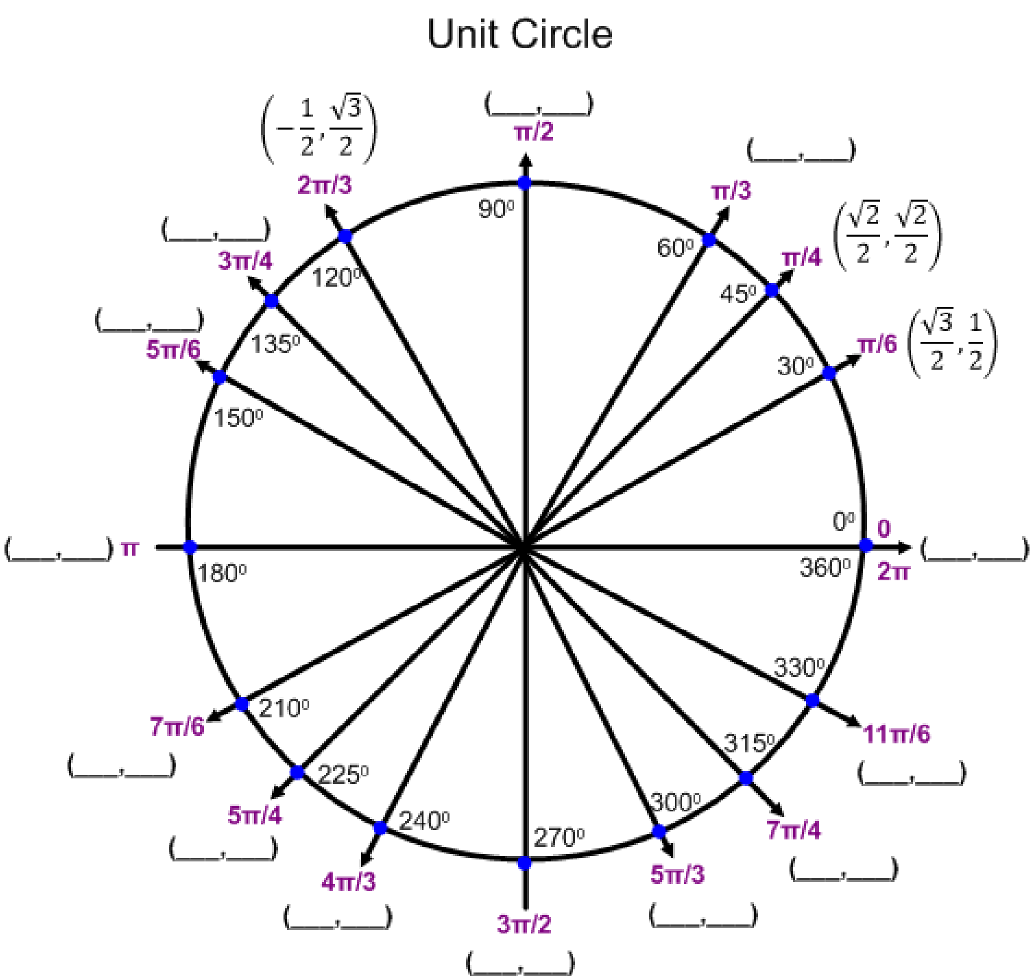
CRITICAL THINKING

1) Explain how to find reference angle if the given angle is located in the fourth quadrant in one sentence.

   Answers may vary.

   If the given angle is in the fourth quadrant, subtract the given angle from 360.

2) Given all the degree and radian, derive all the missing coordinates of the unit circle shown below.



Answers:

$0°$ and $360°=(1,0)$, $60°=\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$, $90°=(0,1)$, $135°=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$, $150°=\left(-\frac{\sqrt{3}}{2},\frac{1}{2}\right)$

$180°=(-1,0)$, $210°=\left(-\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$, $225°=\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$, $240°=\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

$270°=(0,-1)$, $300°=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$, $315°=\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$, $330°=\left(\frac{\sqrt{3}}{2},-\frac{1}{2}\right)$

3) By looking at the unit circle above, notice the $30°$ and $210°$ forms a straight line. Explain their relationship in terms of radian.

   Answers may vary.

   Since $30°$ and $210°$ forms a straight line and a straight line has $180°$ or π radian, thus the relationship in terms of radian will be taking $\frac{\pi}{6}$ radian or $30°$ add to π radian and get $\frac{7\pi}{6}$ radian.

4) What is the similarity or difference between the two coordinates of 150° and 330°?

   Answers may vary.

   The similarity is that both the coordinate has the same value except the signs are different since both angles are located in the different quadrants.

5) Explain how special right triangles and reference angle are used to derive the coordinates of $330°$.

   Answers may vary.

   The concept of reference angle is first used to determine which special right triangle will be used next to derive the coordinate. Since the reference angle of $330°$ is $30°$, use the $30°$-$60°$ special right triangle to find the $\cos 330° = x = \frac{\sqrt{3}}{2}$ and $\sin 330° = y = -\frac{1}{2}$.

ATTRIBUTION TABLE

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